

The Price of Fixed Income Market Volatility

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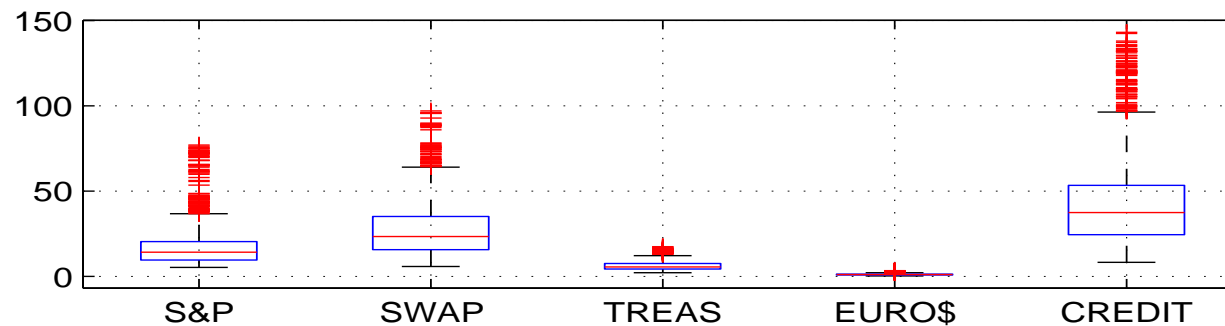
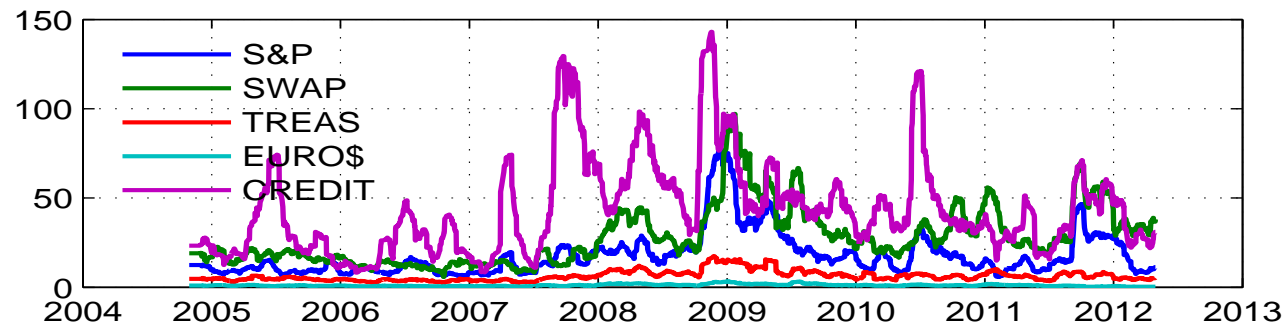
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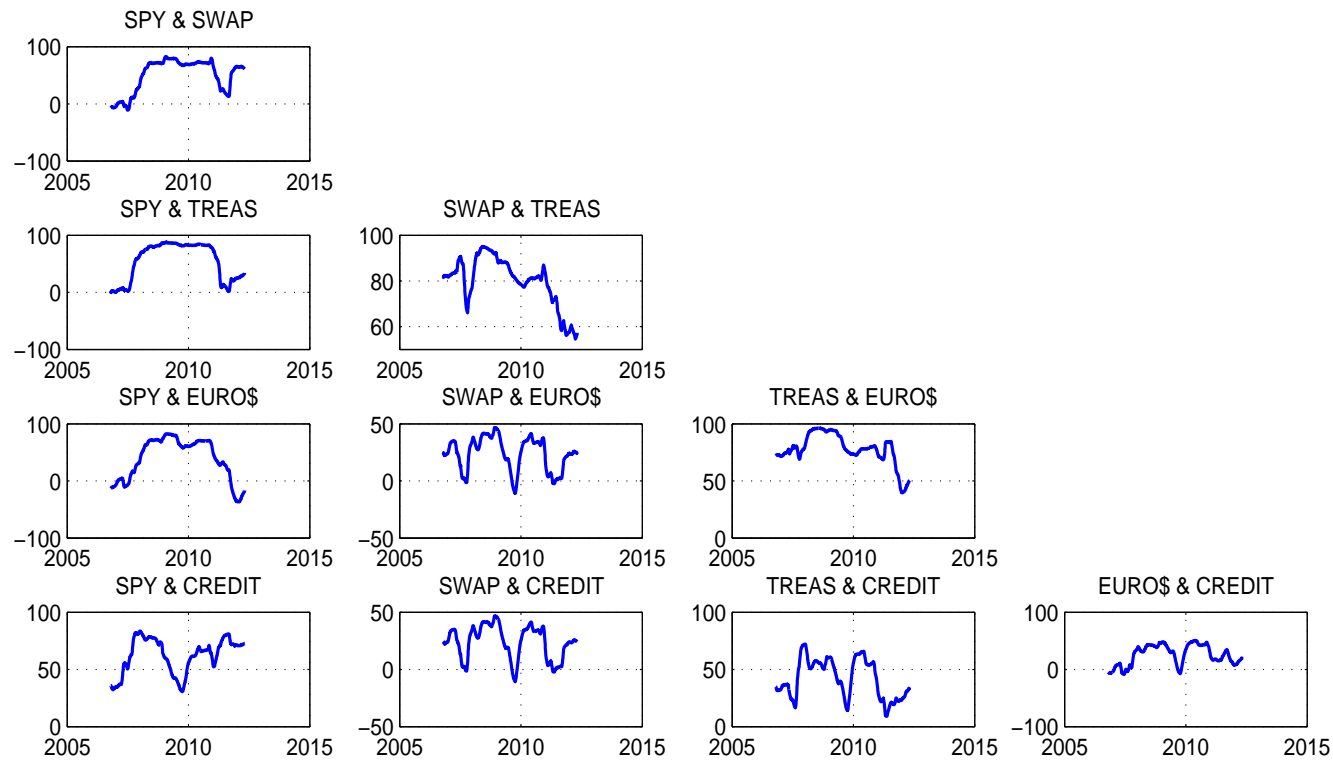
Why fixed income market volatility?

- \$14.5 trillions notional US dollar interest rate option markets
- Hedging interest rate risk could not be more timely over ongoing uncertainties
- Fixed income volatility and equity volatility evolve heterogeneously over time, co-moving disproportionately during periods of global imbalances and each reacting to events of different nature
 - VIX derivatives are not necessarily enough to hedge risks arising in fixed income markets

Realized volatility across equity and fixed income markets



Correlations



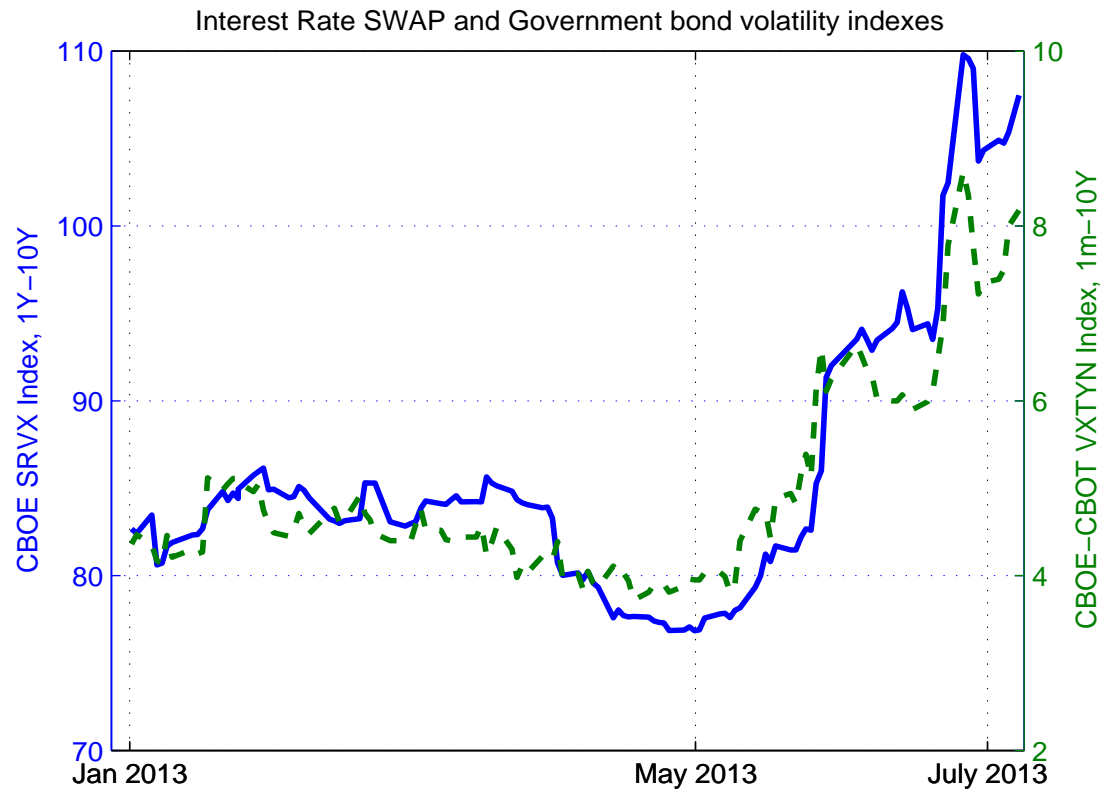
Purpose

- Methodology for options-based “model-free” pricing of equity volatility has been known for some time
 - Little is known about analogous methodologies for pricing various fixed income volatilities
- Aim to provide a unified evaluation framework of fixed income volatility
 - Deal with disparate markets such as interest rate swaps, government bonds, time-deposits and credit
 - Develop model-free, forward looking indexes of fixed income volatility that match different quoting conventions across various markets
 - Uncover subtle yet important pitfalls arising from naïve superimpositions of the standard equity volatility methodology when pricing various fixed income volatilities

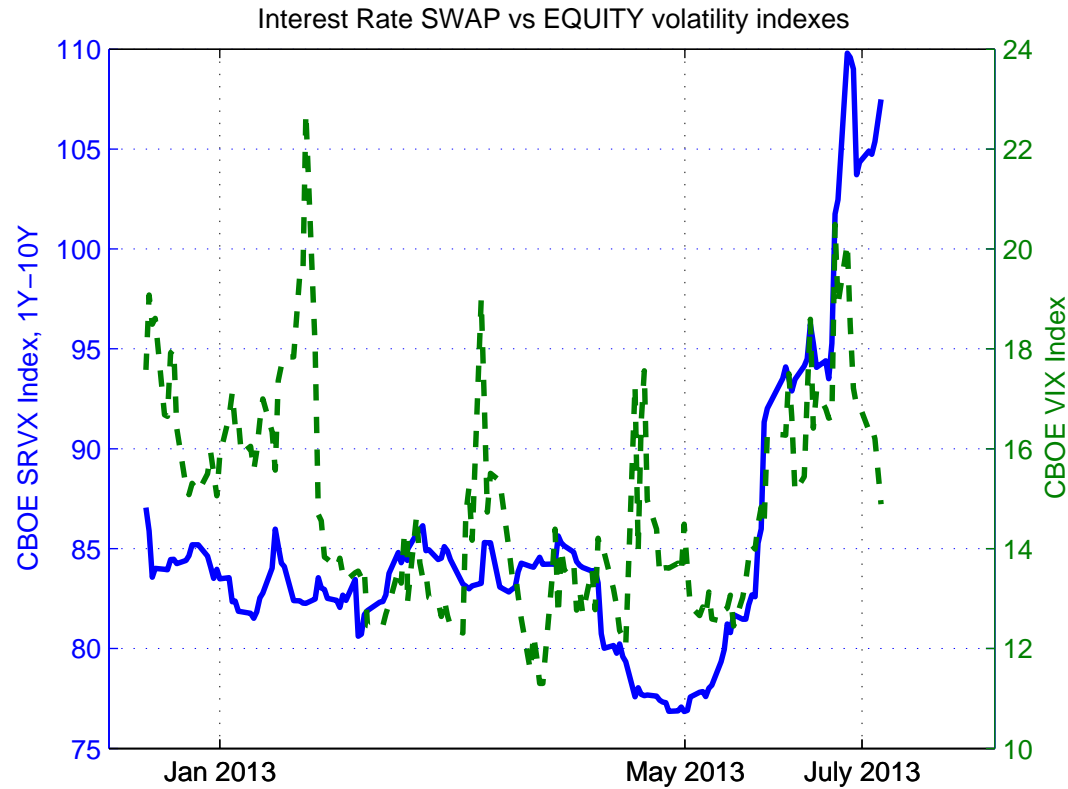
CBOE interest rate volatility indexes

- Some of the interest rate volatility indexes in this book are currently being implemented by the Chicago Board Options Exchange (CBOE)
- Behavior of two recently launched indexes of fixed income volatility that parallel the equity VIX

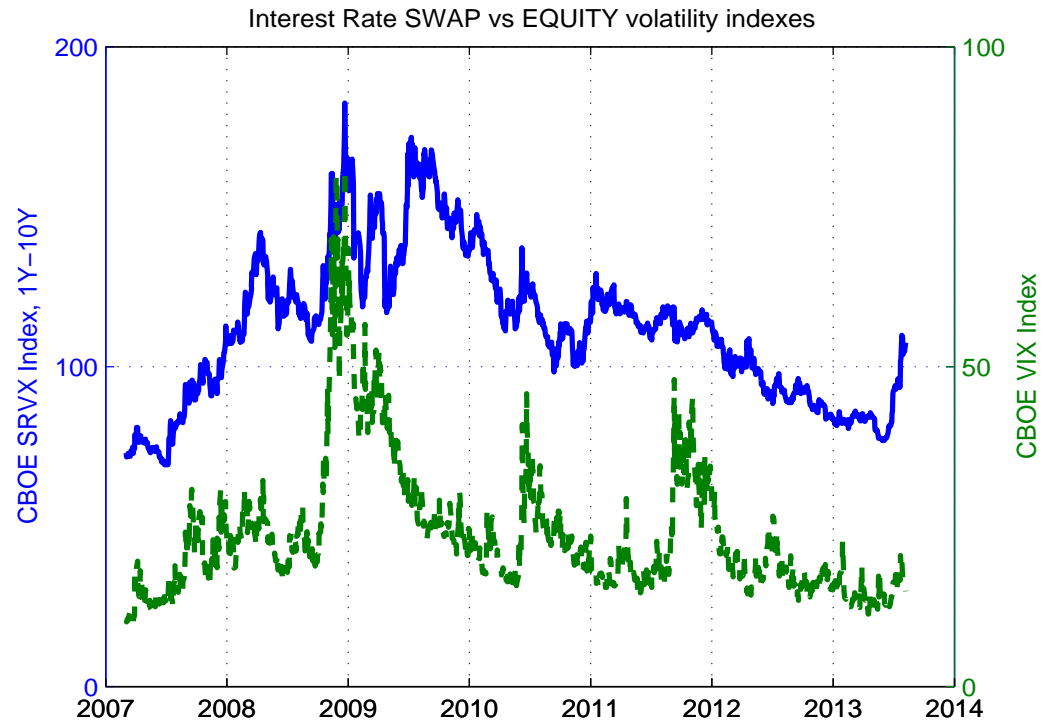
A snapshot



Tete-a-tete, CBOE interest rate volatility & VIX indexes



Tete-a-tete, II



Rates versus Equity: Key issues of methodology

- Extends the work of Carr & Madan (2001) to interest rate and credit sensitive securities
- Rates & Fixed income market numéraires
 - The nature of FI security evaluation involves numéraires beyond the MMA numéraire
- Model-free pricing requires dedicated contract designs
 - Match market practice—Basis point vs logN, price vs rate vols
 - Key insight: link BP variance to Quadratic contracts, and then price Q-contracts à la Carr-Madan

Plan

Part 1: Variance contracts: fixed income security design

Part 2: Interest rate swap markets

Part 3: Government bonds and time deposits

Part 4: Credit

Sources

- Mele, Antonio, and Yoshiki Obayashi, 2013. “The Price of Fixed Income Market Volatility.” Book manuscript (\approx 200 pages)
- Mele, Antonio, and Yoshiki Obayashi, 2012. “An Interest Rate Swap Volatility Index and Contract.” Technical white paper underlying the CBOE Interest Rate Swap Volatility Index. Available from <http://www.cboe.com/micro/srvx/default.aspx>.
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- Mele, Antonio, and Yoshiki Obayashi, 2013. “The Price of Government Bond Volatility.” Swiss Finance Institute Working paper.

- Mele, Antonio, and Yoshiki Obayashi, 2013. “Credit Variance Swaps and Volatility Indexes.” Swiss Finance Institute Working paper.

And

- Mele, Antonio, and Yoshiki Obayashi, US Patent App. 13/528,150: “Methods and Systems for Creating an Interest Rate Swap Volatility Index and Trading Derivative Products Based Thereon.” Patent issued.
- Mele, Antonio, and Yoshiki Obayashi, US Patent App. 13/842,197: “Methods and Systems for Creating a Government Bond Volatility Index and Trading Derivative Products Based Thereon.” Patent pending.
- Mele, Antonio, and Yoshiki Obayashi, US Patent App. 13/842,050: “Methods and Systems for Creating a Time Deposit Volatility Index and Trading

Derivative Products Based Thereon.” Patent pending.

- Mele, Antonio, and Yoshiki Obayashi, US Patent Appl. 13/841,653: “Methods and Systems for Creating a Credit Volatility Index and Trading Derivative Products Based Thereon.” Patent pending.

1/4 Variance contracts: fixed income security design

Variance swaps in fixed income markets: overview of methodology

- Search for:
 - (i) a numéraire in each FI security market, such that all security prices in terms of this numéraire are martingales (*absence of arbitrage*) under Q^N ; and
 - (ii) a numéraire in the variance swap market for the given security such that variance swaps are priced in a model-free fashionNuméraires (i) and (ii) are not necessarily the same, although they could be
- Basis point variance contracts are quite relevant in FI markets, and deserve a special treatment
 - Rely on “Quadratic contracts” as mentioned which we span through Carr-Madan expansions

Additional issues

- How *BP* volatility indexes work compared to *percentage*?
- How truncating calculations to include a finite number of options affect the behavior of *BP* and *percentage* volatility indexes?
- Pitfalls from applying equity methodology to fixed income are explained in Part III

The right numéraire

- Aim to price volatility in a “model-free” fashion
 - “Model-free” —only relies on the price of traded assets (e.g., European-style interest rate derivatives, zero coupon bonds or defaultable bonds)
- Design variance swap contracts with fair value leading to indexes that reflect market expectations of fixed income market volatility, adjusted for the “relevant notion” of market risk

- What is the “relevant notion” of market risk-adjustment in our context?
 - Absence of arbitrage implies there is a unit of account, aka *numéraire*, such that the prices of all securities specified in terms of this unit are martingales under a certain probability—facilitate calculations via Black’s pricers
 - This *numéraire* is the relevant notion of risk-adjustment in our context
- In the equity case, the numéraire is the money market account *assuming interest rates are constant*
- In the FI market, there is a notion of numéraire specific to each market
 - We provide a unifying methodology

Market numéraires and volatilities: definitions

- *Forward starting agreement* originated at t , with delivery of

$$\Pi_T \equiv N_T \times (X_T - K), \quad \text{at } T,$$

where X_T and N_T are measurable wrt information at T , \mathbb{F}_T , and K is chosen at t , such that the value of the contract is zero at inception

- Q is the RN prob, $\mathbb{E}_t(\cdot)$ the expectation under Q conditional on \mathbb{F}_t , r_τ the short-term rate process
- N_τ is the price of a tradeable asset
- Under regularity conditions, there exist (i) a probability Q^N , and (ii) a martingale process X_τ under Q^N that clears the agreement, $X_t = K$

- X_τ is the *forward risk process*, and N_τ is the *market numéraire* at T , such that any asset price process S_t normalized by N_t is a martingale under Q^N ,

$$\frac{S_t}{N_t} = \mathbb{E}_t^{Q^N} \left(\frac{S_T}{N_T} \right),$$

where $\mathbb{E}_t^{Q^N}$ denotes the conditional expectation under Q^N

- Q^N *market numéraire probability*
 - the annuity probability in interest rate swap markets
 - the forward probability in government bond and time deposit markets
 - the defaultable annuity probability in credit markets
 -

- It is the volatility of X_τ that we are interested in pricing
- Take X_t to be a strictly positive diffusion process with stochastic volatility (we consider jumps later),

$$\frac{dX_\tau}{X_\tau} = \sigma_\tau \cdot dW_\tau, \quad \tau \in [t, T]$$

where W_t is a multidimensional Wiener process under Q^N

- Integrated variance over a time interval $[t, T]$. Based on
 - *Arithmetic or basis point* (BP henceforth) changes of X_t ,

$$V^{\text{bp}}(t, T) \equiv \int_t^T X_\tau^2 \|\sigma_\tau\|^2 d\tau$$

- *Logarithmic or percentage* changes of X_t ,

$$V(t, T) \equiv \int_t^T \|\sigma_\tau\|^2 d\tau$$

Example

- Forward starting swap payer with fixed interest K has payoff,

$$\text{Swap}_T(K; T_1, \dots, T_n) = \text{PVBP}_T(T_1, \dots, T_n) [R_T(T_1, \dots, T_n) - K],$$

where $R_t(T_1, \dots, T_n)$ is the fwd swap rate at t for a contract beginning at T_0 and reset dates T_1, \dots, T_n , and $\text{PVBP}_T(T_1, \dots, T_n)$ is the annuity factor

- The annuity probability Q_{sw} has Radon-Nikodym against Q ,

$$\left. \frac{dQ_{\text{sw}}}{dQ} \right|_{\mathbb{F}_T} = e^{-\int_t^T r_s ds} \frac{\text{PVBP}_T(T_1, \dots, T_n)}{\text{PVBP}_t(T_1, \dots, T_n)}$$

- In a diffusion environment,

$$\frac{dR_\tau(T_1, \dots, T_n)}{R_\tau(T_1, \dots, T_n)} = \sigma_\tau \cdot dW_\tau^{\text{sw}}, \quad \tau \in [t, T]$$

where W_τ^{sw} is a Brownian motion under Q_{sw}

Back to general →

Forward contract with stochastic multiplier

- It's the contract with payoff,

$$Y_T \times \left(\Psi \left(\{X_\tau\}_{\tau \in [t, T]} \right) - K_Y \right),$$

where Y_T is \mathbb{F}_T -measurable, Ψ is a functional of the entire path of X , and K_Y is the fair value of the contract,

$$K_Y = E_t^{Q^Y} \left(\Psi \left(\{R_\tau\}_{\tau \in [t, T]} \right) \right), \quad \text{where} \quad \left. \frac{dQ^Y}{dQ} \right|_{\mathbb{F}_T} = \frac{e^{-\int_t^T r_u du} Y_T}{\mathbb{E}_t \left[e^{-\int_t^T r_u du} Y_T \right]}$$

Refer to Q^Y as the forward multiplier probability

- **Example** Basis point variance contracts: $\Psi \left(\{R_\tau\}_{\tau \in [t, T]} \right) = V^{\text{bp}}(t, T)$

Model-free pricing

- **Definition:** K_Y is model-free if we can find a numéraire N_T and a contract multiplier Y_T such that K_Y equals the value of a portfolio of European call and option prices with strike K , say $\text{Call}_t(K)$ and $\text{Put}_t(K)$, where:

$$\frac{\text{Call}_t(K)}{N_t} = \mathbb{E}_t^{Q^N} \left(\frac{\max\{\Pi_T, 0\}}{N_T} \right), \quad \frac{\text{Put}_t(K)}{N_t} = \mathbb{E}_t^{Q^N} \left(\frac{\max\{-\Pi_T, 0\}}{N_T} \right)$$

Model-free contracts

- **Proposition** K_Y is model-free if and only if the Radon-Nikodym derivative of the fwd multiplier prob Q^Y against the mkt numéraire prob Q^N , is uncorrelated with $V^{\text{bp}}(t, T)$ and $V(t, T)$

For $\Psi(\cdot) = V^{\text{bp}}(t, T)$ (Basis Point variance pricing) it is given by:

$$K_Y = \frac{2}{N_t} \left(\int_0^{X_t} \text{Put}_t(K) dK + \int_{X_t}^{\infty} \text{Call}_t(K) dK \right)$$

For $\Psi(\cdot) = V(t, T)$ (Percentage variance pricing) it is given by:

$$K_Y = \frac{2}{N_t} \left(\int_0^{X_t} \frac{\text{Put}_t(K)}{K^2} dK + \int_{X_t}^{\infty} \frac{\text{Call}_t(K)}{K^2} dK \right)$$

Remarks

- (i) Suggest choices for the random multiplier Y_T in each context of interest
- Equity & constant interest rates: $N_t = e^{-\bar{r}(T-t)}$, $Y_T = 1$
 - Fixed income: necessary and sufficient conditions under which variance swaps are model-free. Most intuitive case

$$Y_T = N_T$$

(stochastic contract multiplier = market numéraire)

- (ii) We are not merely re-stating that in the absence of arbitrage, security prices rescaled by some N_t are martingales under Q^N
- Saying something stronger: variance swaps are *model-free* if their payoffs are re-scaled by the market numéraire

(iii) The situation is actually intricate—Proposition identifies a host of possible variance swap “tilters”

- Consider the following example of stochastic multiplier

$$Y_T = N_T \epsilon_T,$$

where ϵ_T is any \mathbb{F}_T -measurable random variable satisfying $\text{cov}^{Q^N} [V^{\text{bp}}(t, T), \epsilon_T] = 0$

- However, the market numéraire is the benchmark of this book because of its economic appeal and the familiarity with it by academics and practitioners

(iv) Finally, we check the internal consistency of our contract design

Consider highly idealized case of a Gaussian model,

$$dX_\tau = \sigma_n \cdot dW_\tau$$

Obvious counterpart to the standard Black-Scholes. We have,

$$K_Y = \frac{2}{N_t} \left(\int_0^{X_t} \text{Put}_t(K) dK + \int_{X_t}^{\infty} \text{Call}_t(K) dK \right) = \|\sigma_n\|^2 (T - t).$$

Log versus quadratic contracts

- We hinge on a key and novel insight—the price of a BP variance swap links to that of a “quadratic contract,”

$$X_T^2 - X_t^2 = V^{\text{bp}}(t, T) + 2 \int_t^T X_\tau dX_\tau,$$

such that,

$$\mathbb{E}_t^{Q^N}(V^{\text{bp}}(t, T)) = \mathbb{E}_t^{Q^N}(X_T^2 - X_t^2)$$

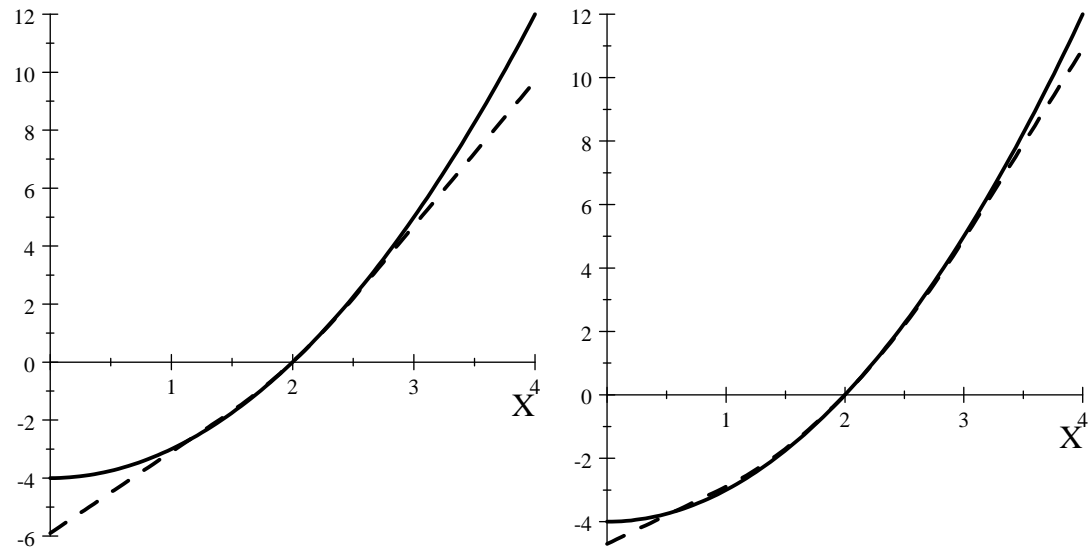
- Radically new hedging implications

- Payoff of a quadratic contract is roughly the sum of (i) two forwards, and (ii) two portfolios comprising OTM and one ATM option,

$$X_T^2 - X_o^2 \approx 2X_o(X_T - X_o) + 2 \left(\sum_{j:K_j < X_o} (K_j - X_T)^+ + \sum_{j:K_j \geq X_o} (X_T - K_j)^+ \right) \Delta K$$

- In contrast, the payoff of a logarithmic contract can be approximated as,

$$\ln \frac{X_T}{X_o} \approx \frac{1}{X_o} (X_T - X_o) - 2 \left(\sum_{j:K_j < X_o} \frac{1}{K_j^2} (K_j - X_T)^+ + \sum_{j:K_j \geq X_o} \frac{1}{K_j^2} (X_T - K_j)^+ \right) \Delta K$$



Hedging quadratic contracts with options. Solid lines Quadratic contract, $X^2 - X_o^2$, with $X_o = 2$. Dashed lines Replicating portfolios comprising: (i) two forwards struck at $X_o = 2$; and (ii) two additional equally weighted portfolios, with $\Delta K = \frac{1}{10}$, each including one ATM option, and 10 OTM options (left panel) or 20 OTM options (right panel)

Volatility indexes

- Model-free indexes of expected volatility,

$$VX_t^j(T) \equiv \sqrt{(T-t)^{-1} V_t^j}, \quad j \in \{\text{bp}, \text{p}\},$$

where V_t^{bp} is the strike K_Y for the basis point variance contract, and V_t^{p} is the strike K_Y for the percentage contract

- Index decompositions Assume “sticky smile,” i.e. $\sigma(X, K) = \sigma(\lambda X, \lambda K)$, $\lambda > 0$. Then,

- there exists a function $\xi(t, T)$ independent of X , such that $V_t^{\text{bp}} = X_t^2 \times \xi(t, T)$
- V_t^{p} is independent of X

Skew shifts and the dynamics of volatility indexes

- Truncations In practice, rely on a finite number (/bounded set) of options. Suppose

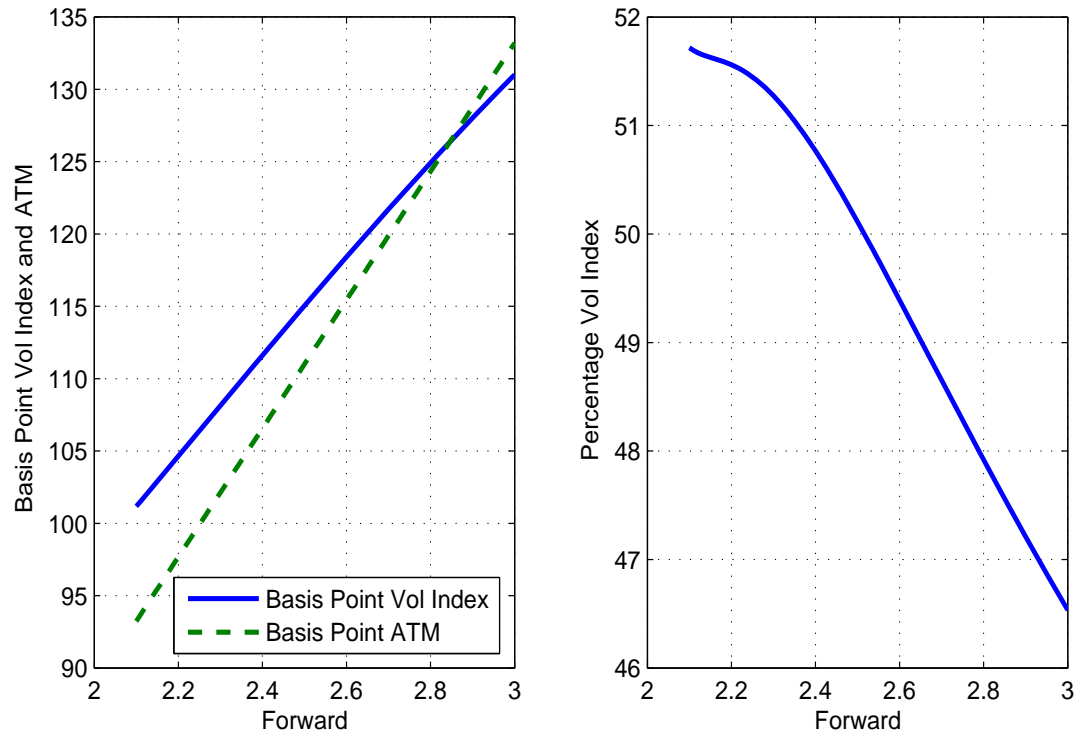
$$V_{\ell}^{\text{bp}} \equiv \int_{X-\ell}^X \text{Put}(X, K, \sigma(X, K)) dK + \int_X^{X+\ell} \text{Call}(X, K, \sigma(X, K)) dK$$

and

$$V_{\ell} \equiv \int_{X-\ell}^X \frac{\text{Put}(X, K, \sigma(X, K))}{K^2} dK + \int_X^{X+\ell} \frac{\text{Call}(X, K, \sigma(X, K))}{K^2} dK$$

- Provide theoretical study \longrightarrow Volatility indexes might respond to movements in the forward, possibly going beyond those relating to the fundamentals

Numerical example: Skew is constant but indexes change



Left panel: BP expected volatility vs BP ATM volatility. Right panel: Percentage expected volatility.

Resilience to jumps

- Assume

$$\frac{dX_\tau}{X_\tau} = - \left(\mathbb{E}_\tau^{Q^N} \left(e^{j(\tau)} - 1 \right) \eta(\tau) \right) d\tau + \sigma_\tau \cdot dW(\tau) + \left(e^{j(\tau)} - 1 \right) dJ(\tau),$$

where $J(\tau)$ is a Cox process under Q^N with intensity equal to $\eta(\tau)$, and $j(\tau)$ is the logarithmic jump size.

- Indexes are,

$$VX_{J_t}^{\text{bp}}(T) = VX_t^{\text{bp}}(T) \text{ (basis point)}$$

$$K_{J,Y} \equiv K_Y - 2\mathbb{E}_t^{Q^N} \left[\int_t^T \left(e^{j(\tau)} - 1 - j(\tau) - \frac{1}{2}j^2(\tau) \right) dJ(\tau) \right] \text{ (perc.)}$$

- Example: jumps distribution collapses to $\bar{j} < 0$ with constant intensity $\bar{\eta}$.
Then,

$$VX_{Jt}^P(T) = \sqrt{VX_t^P(T) + 2\bar{\eta}\mathcal{J}}, \quad \mathcal{J} \equiv -(e^{\bar{j}} - 1 - \bar{j} - \frac{1}{2}\bar{j}^2) > 0$$

2/4 Interest rate swap markets

Highlights

- CBOE-SRVX
- Match market practice—Basis point
 - Key insight in the previous part: link BP variance to Quadratic contracts, and then price Q-contracts à la Carr-Madan
- Variance swap hedging: a component of it involves replicating $\int \frac{dR_t}{R_t}$
 - While $\frac{dR_t}{R_t}$ is not a return from any traded asset, the forward swap rate R_t can be dynamically replicated

The dynamics of the forward swap rate

- **Notation**

- Let $R_t(T_1, \dots, T_n)$ be the *Forward Swap Rate* prevailing at t , i.e. the fixed rate such that the value of a forward starting swap (at $T \equiv T_0$, with reset dates T_0, \dots, T_{n-1} , tenor length $T_n - T$, and payment periods $T_1 - T_0, \dots, T_n - T_{n-1}$) is zero at t

- **Assumption**

- We assume $R_t(T_1, \dots, T_n)$ is a *diffusion*,

$$dR_\tau(T_1, \dots, T_n) = R_\tau(T_1, \dots, T_n) \sigma_\tau(T_1, \dots, T_n) \cdot dW_\tau, \quad \tau \in [t, T],$$

where W_τ is a Brownian Motion under the annuity probability, and $\sigma_\tau(T_1, \dots, T_n)$ is adapted to W_τ

- **Basis Point variance**

- Define $V_n^{\text{BP}}(t, T)$, as the *Basis Point* realized variance of the forward swap rate arithmetic changes in the time interval $[t, T]$,

$$V_n^{\text{BP}}(t, T) \equiv \int_t^T R_\tau^2(T_1, \dots, T_n) \|\sigma_\tau(T_1, \dots, T_n)\|^2 d\tau$$

- **Extension to Jumps**

- Basis Point variance is:

$$V_n^{J,\text{BP}}(t, T) \equiv V_n^{\text{BP}}(t, T) + \int_t^T R_\tau^2(T_1, \dots, T_n) \left(e^{j_n(\tau)} - 1\right)^2 dN_\tau,$$

where N_τ is a Cox process under the annuity probability with intensity equal to η_τ , and $j_n(\tau)$ is the logarithmic jump size

- **Jumps irrelevance results:** The price of BP variance is resilient to the presence of jumps as explained in Part I

Swap transaction risks

- **The annuity factor, or the PVBP**

- The value of a fixed rate payer swap is:

$$\text{SWAP}_T(K; T_1, \dots, T_n) = \text{PVBP}_T(T_1, \dots, T_n) [R_T(T_1, \dots, T_n) - K],$$

- **Swaptions**

- Payer's and receiver's payoffs, $\text{PVBP}_T(T_1, \dots, T_n) [R_T(T_1, \dots, T_n) - K]^+$ and $\text{PVBP}_T(T_1, \dots, T_n) [K - R_T(T_1, \dots, T_n)]^+$

- Options on equities relate to a single source of risk, the stock price. Swaps (and swaptions), instead, are tied to two sources of risk:

- the forward swap rate
- the swap's PVBP realized at time T

This adds complexity to defining and pricing swap market volatility

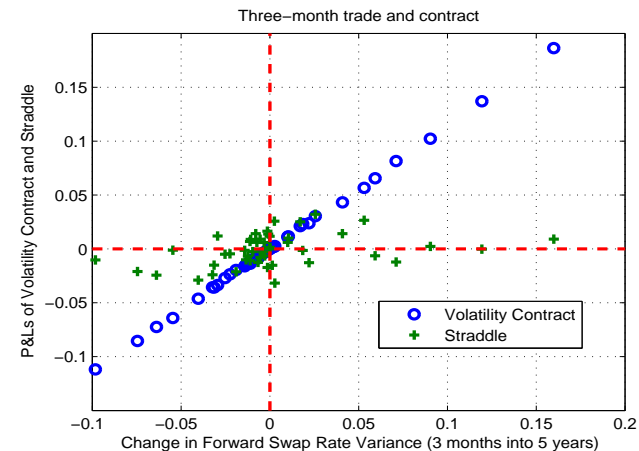
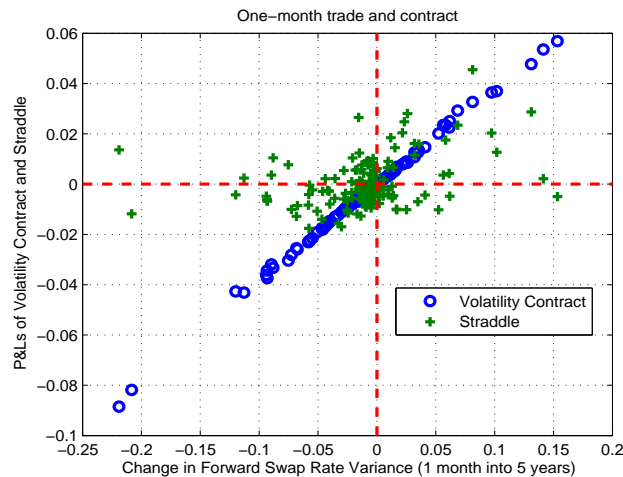
Option-based volatility trading

- Affected by both **price dependency** and the randomness of the **annuity factor**
- Noticed previously by banks while structuring notes linked to interest rate volatility → addressed in an ad hoc way
- **Theoretically,**

$$P\&L_T^{\text{straddle}} \approx \sum_{t=1}^T \Gamma_t^{\$} \cdot [(\sigma_t^2 - IV_0^2) PVBP_T] + \sum_{t=1}^T \text{Straddle}_t \cdot \text{Vol}_t (PVBP_t) \cdot \frac{\widetilde{\Delta R}_t}{R_t}$$

- First term: familiar price-dependency term (El Karoui, Jeanblanc-Picqué and Shreve, 1998)
- Second: new. Links to randomness of the $PVBP_t$ and the shocks affecting the swap rate, $\frac{\widetilde{\Delta R}_t}{R_t}$

- An empirical experiment



- Straddles P&L has the same sign as the variance risk-premium 62% of the time for one-month trades, and 65% of the time for three-month trades
- $\text{Corr}(\text{straddle P\&L, variance risk-premium}) = 33\%$ (one-month), and 28% (three-month)
- When delta-hedged, these trades lead to higher correlations—with average corrs equal to 63% (Jiang, 2011)

Use of methodology in Part 1

- **We use the PVBP as a multiplier** as it is appealing and easy to calculate as explained

Contract design — Three "Gaussian" contracts

- **I: Interest Rate Variance (IRV) Forward Agreement**

At time t , counterparty A promises to pay B the annuity-factor adjusted BP variance realized over $[t, T]$, i.e.

$$V_n^{\text{BP}}(t, T) \times \text{PVBP}_T(T_1, \dots, T_n).$$

The price B shall pay A at time t is the *Interest Rate Variance (IRV) Forward Rate*, and is denoted as $\mathbb{F}_{\text{var},n}(t, T)$

- Links to Constant Maturity Swaps—see below
- Useful for implementing variance trading strategies—see Chapter 3 of the book + CBOE technical white paper

- **II: IRV Swap**

Counterparty A agrees to pay B the following difference at time T :

$$\text{Var-Swap}_n(t, T) \equiv V_n^{\text{BP}}(t, T) \times \text{PVBP}_T(T_1, \dots, T_n) - \mathbb{P}_{\text{var},n}(t, T), \quad (1)$$

where $\mathbb{P}_{\text{var},n}(t, T)$, the *IRV Swap Rate*, is a fixed variance swap rate determined at time t , which makes the current value of $\text{Var-Swap}_n(t, T)$ in Eq. (1) equal to zero

- **III: Standardized IRV Swap**

Counterparty A agrees to pay B the following difference at time T :

$$\text{Var-Swap}_n^*(t, T) \equiv [V_n^{\text{BP}}(t, T) - \mathbb{P}_{\text{var},n}^*(t, T)] \times \text{PVBP}_T(T_1, \dots, T_n), \quad (2)$$

where $\mathbb{P}_{\text{var},n}^*(t, T)$, the *Standardized IRV Swap Rate*, is a fixed variance swap rate determined at t , which makes the current value of $\text{Var-Swap}_n^*(t, T)$ in Eq. (2) equal to zero

- CBOE-SRVXSM is based on $\mathbb{P}_{\text{var},n}^*(t, T)$

Pricing

- One approximation to $\mathbb{F}_{\text{var},n}(t, T)$ based on a finite number of swaptions,

$$\mathbb{F}_{\text{var},n}(t, T) = 2 \left[\sum_{i:K_i < R_t} \text{SWPN}_t^R(K_i, T; T_n) \Delta K_i + \sum_{i:K_i \geq R_t} \text{SWPN}_t^P(K_i, T; T_n) \Delta K_i \right],$$

and,

$$\mathbb{P}_{\text{var},n}(t, T) = \frac{\mathbb{F}_{\text{var},n}(t, T)}{P_t(T)}, \quad \text{and} \quad \mathbb{P}_{\text{var},n}^*(t, T) = \frac{\mathbb{F}_{\text{var},n}(t, T)}{\text{PVBP}_t(T_1, \dots, T_n)}$$

- **BP Index,**

$$\text{Index}_t(T, n) = \sqrt{\frac{1}{T-t} \mathbb{P}_{\text{var},n}^*(t, T)}, \quad \text{CBOE-SRVX}^{\text{SM}} = \text{Index}_t(1Y, 10Y)$$

- Marking to market expressions are in the book

Constant weightings, Gamma exposure interpretation

- *Equity* options portfolio vega is insensitive to stock price with $\frac{1}{K^2}$ weightings in a lognormal market (Demeterfi, Derman, Kamal and Zou, 1999).
- The BP swap counterpart to equity is this:

- Gaussian market,

$$dR_\tau (T_1, \dots, T_n) = \sigma_n \cdot dW_\tau$$

- OTM & ATM swaptions portfolio with weightings $\omega(K)$ and value,

$$\pi_t (R_t, T, \sigma_n) \equiv \int \omega(K) \mathcal{O}_t (R_t, K, T, \sigma_n) dK$$

- Portfolio vega is insensitive to forward swap rate if and only if the weightings are independent of K ,

$$\frac{\partial}{\partial R} \left(\frac{\partial \pi_t (R, T, \sigma)}{\partial \sigma} \right) = 0 \iff \omega(K) = \text{const.} \iff \text{Const. } \Gamma \text{ exposure}$$

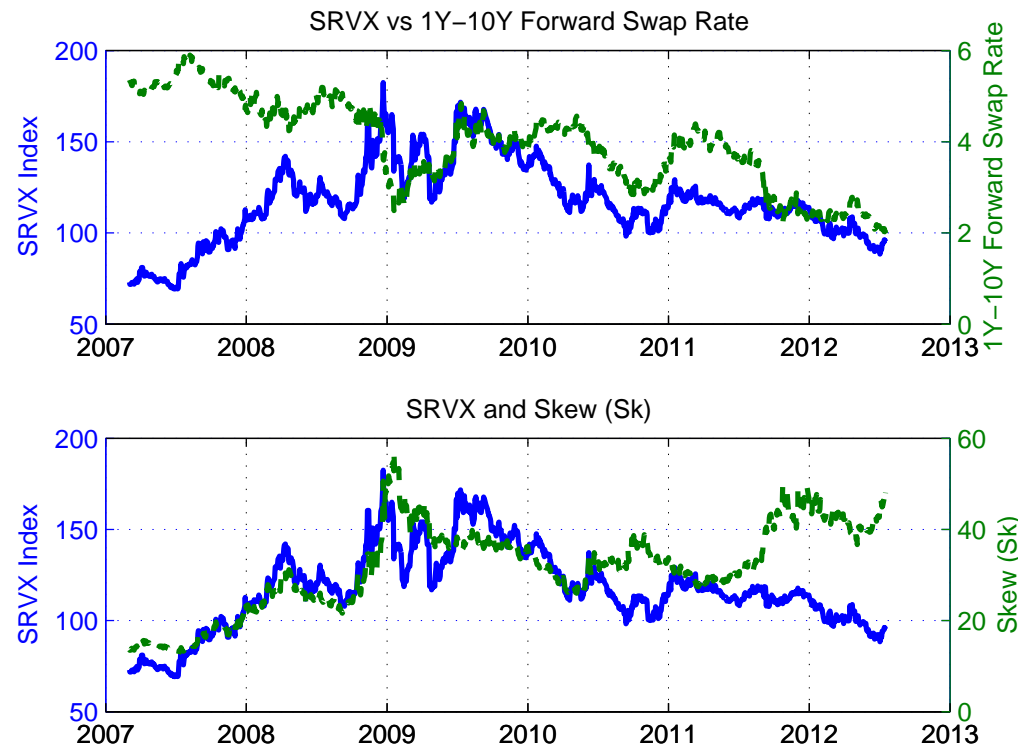
Homogeneity

- Part I: *Suppose sticky smile. Then, $\exists \xi_n(t, T)$ independent of R_t s.t.,*

$$Index_t(T, n) = R_t \times \sqrt{\frac{1}{T-t} \xi_n(t, T)}$$

- Useful to interpret historical behavior of the index (Mele, Obayashi and Shalen, 2013)

R and skew



Hedging variance swaps

- Replicating BP-denominated contracts requires positioning in *quadratic contracts*, i.e. those delivering $R_T^2 - R_t^2$, rescaled by the PVBP.
- Consider, e.g., the *Standardized IRV Swap*.

Replication of the payoff relating to the Standardized BP-IRV contract. ZCB and OTM stand for zero coupon bonds and out-of-the-money

Portfolio	Value at t	Value at T
(i) short self-financed portfolio of ZCB	0	$[V_n^{\text{BP}}(t, T) - (R_T^2 - R_t^2)] \times \text{PVBP}_T$
(ii) long swaps and long OTM swaptions	$-\mathbb{F}_{\text{var},n}(t, T)$	$(R_T^2 - R_t^2) \times \text{PVBP}_T$
(iii) borrow basket of ZCB for $\mathbb{P}_{\text{var},n}^*(t, T) \times \text{PVBP}_t$	$+\mathbb{F}_{\text{var},n}(t, T)$	$-\mathbb{P}_{\text{var},n}^*(t, T) \times \text{PVBP}_T$
Net cash flows	0	$[V_n(t, T) - \mathbb{P}_{\text{var},n}^*(t, T)] \times \text{PVBP}_T$

Links to Constant Maturity Swaps

- CMS are known to relate to the entire swaption skew since at least Hagan (2003) and Mercurio and Pallavicini (2006).
- We make a further step: CMS are actually a basket of BP-IRV forwards
- **Notation.** A party pays a counterparty the *spot* swap rate with a fixed tenor over a sequence of dates with legs set in advance, i.e. $R_{T_0+j\kappa}(T_1 + j\kappa, \dots, T_n + j\kappa)$ at times $T_0 + (j + 1)\kappa$, $j = 0, \dots, N - 1$,
 - Approximately, the price of a CMS is:

$$\begin{aligned} \text{CMS}_N(t) &\equiv \sum_{j=0}^{N-1} P_t(\tau_j + \kappa) R_t(T_1 + j\kappa, \dots, T_n + j\kappa) \\ &\quad + \sum_{j=0}^{N-1} G'(R_t(T_1 + j\kappa, \dots, T_n + j\kappa)) \mathbb{F}_{\text{var},n}(t, T_0 + j\kappa), \quad \text{for some function } G(\cdot) \end{aligned}$$

- So IR variance swaps can be used to hedge against CMS

3/4 Government bonds and time-deposits

The numéraire

- Government bonds (GB) and time deposits (TD) share the same *numéraire*, a zero coupon bond, such that
 - the market probability is the forward probability,

$$\left. \frac{dQ_{FT}}{dQ} \right|_{\mathbb{F}_T} = \frac{e^{-\int_t^T r_s ds}}{P_t(T)}$$

- Nevertheless GB & TD variance swaps have distinct features justifying separate treatments
- Recently launched CBOE/CBOT VXTYN on 10 Treas note vol

Begin with GB →

Pitfalls arising while applying standard VIX methodology to rates

Two introductory counter-examples

(i) No-spanning

(ii) Cash options

- We'll provide a third counter-example at the end of this part (“maturity mismatch”)

Example 1—No-spanning

- Price of a zero expiring at T ,

$$\frac{dP_\tau(T)}{P_\tau(T)} = r_\tau d\tau + \sigma_\tau(T) \cdot dW_\tau, \quad t \leq \tau \leq T, \quad (1)$$

where W_τ is a multid. BM under Q

- Variance swap has payoff that *occurs precisely at T* ,

$$\int_t^T \|\sigma_\tau(T)\|^2 d\tau - \mathbb{P}(t, T),$$

where $\mathbb{P}(t, T)$ is the fair value of the strike

- Fair value is:

$$\begin{aligned}\mathbb{P}(t, T) &= \frac{1}{P_t(T)} \mathbb{E}_t \left(e^{-\int_t^T r_\tau d\tau} \int_t^T \|\sigma_\tau(T)\|^2 d\tau \right) \\ &= -2\mathcal{R}_{t,T} + \frac{2}{P_t(T)} \mathbb{E}_t \left(e^{-\int_t^T r_\tau d\tau} \int_t^T \frac{dP_\tau(T)}{P_\tau(T)} \right),\end{aligned}$$

where the log-return on the zero is

$$\mathcal{R}_{t,T} \equiv -\ln P_t(T) = \frac{1 - P_t(T)}{P_t(T)} - \int_{P_t(T)}^1 (1 - K) \frac{1}{K^2} dK$$

- It isn't model-free because $P_\tau(T)$ is not a martingale under the forward prob Q_{FT} —naturally, the forward price *is*
- The “spanned part,” $-2\mathcal{R}_{t,T}$, is actually negative so the unspanned part has a lot of economic meaning but it's model-dependent

Example 2—Cash options

- Suppose we are given a number of quotes re: ATM and OTM bond options settled on cash, not futures, as it is customary in OTC markets
- Aggregate these option prices following standard equity methodology to create a VIX-like index
- Would the resulting index necessarily reflect the fair value of a variance swap?
 - The answer is in the negative

- We have

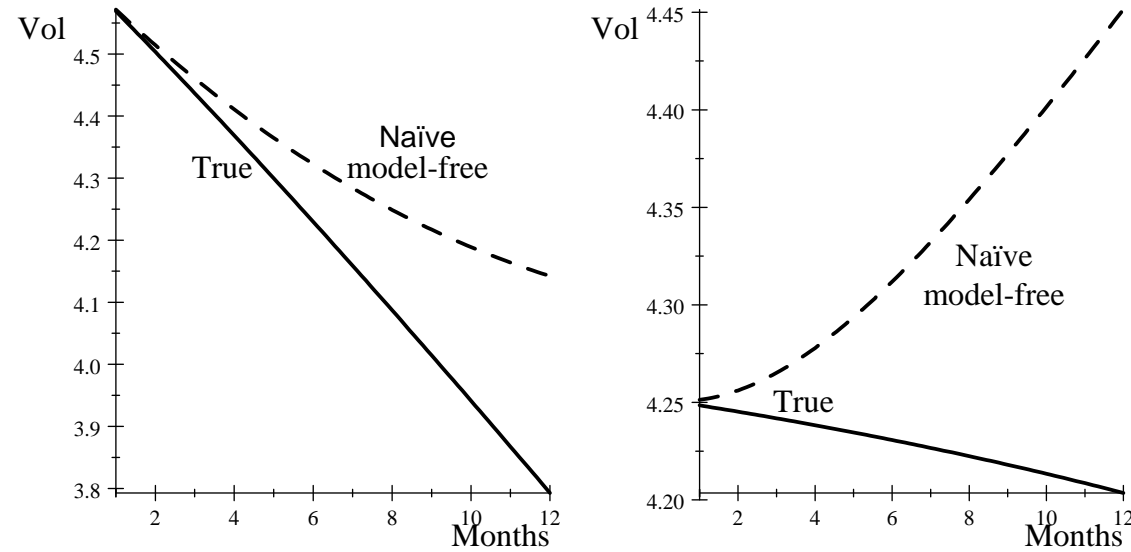
$$\mathbb{P}(t, T, \mathbb{T}) = \mathbb{P}_{\text{VIX}}(t, T, \mathbb{T}) + \text{Bias}(t, T),$$

where

$$\begin{aligned} & \mathbb{P}_{\text{VIX}}(t, T, \mathbb{T}) \\ & \equiv \frac{2}{P_t(T)} \left(\int_0^{F_t^o(T, \mathbb{T})} \text{Put}_t(K, T, \mathbb{T}) \frac{1}{K^2} dK + \int_{F_t^o(T, \mathbb{T})}^{\infty} \text{Call}_t(K, T, \mathbb{T}) \frac{1}{K^2} dK \right) \end{aligned}$$

$$\text{Bias}(t, T) \equiv 2 \int_t^T \left[\mathbb{E}_t^{Q_{FT}}(r_\tau) - \mathbb{E}_t^{Q_{F\tau}}(r_\tau) \right] d\tau$$

and $F_t^o(T, \mathbb{T}) \equiv \frac{P_t(\mathbb{T})}{P_t(T)}$ is the “shadow price” of a forward contract



Biases arising while estimating expected volatility with standard model-free methodology in a hypothetical Vasicek's (1977) market (right panel obtained with a lower persistence than the left)

Basis asset

- Coupon bearing bond issued at time T_0 , and paying off coupons $\frac{C_i}{n}$ over T_i , $i = 1, \dots, N$, $\mathbb{T} \equiv T_N$, where n is the frequency of coupon payments, and $T_i - T_{i-1} = \frac{1}{n}$
- Price of coupon bearing bond is $B_t(\mathbb{T})$. Forward bond price is $F_t(T, \mathbb{T}) = \frac{B_t(\mathbb{T})}{P_t(T)}$, and satisfies,

$$\frac{dF_\tau(T, \mathbb{T})}{F_\tau(T, \mathbb{T})} = v_\tau(T, \mathbb{T}) \cdot dW_{FT}(\tau), \quad \tau \in (t, T),$$

where $W_{FT}(\tau)$ is BM under the fwd prob, $v_\tau(T, \mathbb{T}) \equiv \sigma_\tau^B(\mathbb{T}) - \sigma_\tau(T)$, and $\sigma_\tau^B(\mathbb{T})$ is the vector of inst. vols of the coupon bearing bond

Pricing

- Apply methodology in Part I to obtain,

– Percentage

$$\text{GB-VI}(t, T, \mathbb{T}) \equiv 100 \times \sqrt{\frac{\mathbb{P}(t, T, \mathbb{T})}{T - t}},$$

where $\mathbb{P}(t, T, \mathbb{T})$ is the fair value of a perc. variance swap,

$$\begin{aligned} \mathbb{P}(t, T, \mathbb{T}) &= \frac{2}{P_t(T)} \left(\int_0^{F_t(T, \mathbb{T})} \text{Put}_t(K, T, \mathbb{T}) \frac{1}{K^2} dK + \int_{F_t(T, \mathbb{T})}^{\infty} \text{Call}_t(K, T, \mathbb{T}) \frac{1}{K^2} dK \right) \end{aligned}$$

– Basis point

$$\text{GB-VI}^{\text{bp}}(t, T, \mathbb{T}) \equiv 100 \times 100 \times \sqrt{\frac{\mathbb{P}^{\text{bp}}(t, T, \mathbb{T})}{T - t}},$$

where $\mathbb{P}^{\text{bp}}(t, T, \mathbb{T})$ is the fair value of a BP variance swap,

$$\begin{aligned} & \mathbb{P}^{\text{bp}}(t, T, \mathbb{T}) \\ &= \frac{2}{P_t(T)} \left(\int_0^{F_t(T, \mathbb{T})} \text{Put}_t(K, T, \mathbb{T}) dK + \int_{F_t(T, \mathbb{T})}^{\infty} \text{Call}_t(K, T, \mathbb{T}) dK \right) \end{aligned}$$

Getting the right volatility with the wrong model?

- Consider the following stochastic volatility extension of the Ho and Lee (1986) model,

$$\begin{cases} dr_{\tau} &= \theta_{\tau}d\tau + v_{\tau}dW_1(\tau) \\ dv_{\tau}^2 &= \xi v_{\tau}dW_2(\tau) \end{cases}$$

where ξ is a “volatility of variance” parameter, θ_{τ} is an “infinite-dimensional” parameter we need, to fit the initial yield curve at $\tau = t$ without error, and W_i are BM under the RN prob

- Zero coupon bond price is,

$$P_{\tau}(r_{\tau}, v_{\tau}^2, T) \equiv e^{-\int_{\tau}^T (s - T) \theta_s ds - (T - \tau) r_{\tau} + C_T(\tau) v_{\tau}^2},$$

where

$$\theta_\tau = \frac{\partial f_\$(t, \tau)}{\partial \tau} + \frac{\partial^2 C_\tau(t)}{\partial \tau^2} v_t^2,$$

and $C_T(\tau)$ is the solution to Riccati's equation,

$$\dot{C}_T(\tau) = -\frac{1}{2}(T - \tau)^2 - \frac{1}{2}\xi^2 C_T^2(\tau), \quad C_T(T) = 0$$

- In this market, the fair value of a GB variance swap is (the square of),

$$\text{GB-VI}(v_t; t, T, \mathbb{T}) \equiv \sqrt{\frac{1}{T-t} \int_t^T \bar{\phi}_\tau(t, T, \mathbb{T}) d\tau} \cdot v_t,$$

for some deterministic function $\bar{\phi}_\tau(t, T, \mathbb{T})$

- Under the misspecified assumption that the same GB variance swap is priced with the “wrong probability,” the risk-neutral,

$$\text{GB-VI}(\widehat{v_t}; t, T, \mathbb{T}) \equiv \sqrt{\frac{1}{T-t} \int_t^T \phi_\tau(T, \mathbb{T}) d\tau} \cdot v_t,$$

for some deterministic function $\phi_\tau(T, \mathbb{T}) \neq \bar{\phi}_\tau(t, T, \mathbb{T})$.

- It's an arbitrage in a frictionless capital market
- Consider for example a future of squared index
 - Theoretically future has the same price as the square index, $\text{GB-VI}(v_t; t, T, \mathbb{T})$, because v_τ^2 is a martingale
 - But according to $\text{GB-VI}(\widehat{v_t}; t, T, \mathbb{T})$, there might be contango or backwardation according to parameter values

Basis point / duration-based yield volatility

- It is market practice to publish measures of “basis point yield volatility”
 - The level of volatility of yields consistent with a given ATM option price
 - This measure is model-dependent by nature
 - For example, you may identify yield volatility with the amount of a parallel shift in the yield curve, required to make a pricer consistent with the given option price
 - Model dependency relates to the assumption of a parallel shift in the yield curve, or a weaker assumption of a shift predicted by a multifactor model
- Propose a model-free approach to measure basis point yield volatility
- We rely on a notion of “certainty equivalent prices”

Certainty equivalent prices

- What is the guaranteed price of the coupon bearing bond at time T , say $\mathcal{B}(t, T, \mathbb{T})$, such that the BP volatility index is the same as an hypothetical BP volatility index in this certainty equivalent market?
- Clearly, such an hypothetical market is also one where the forward is constant and equal to $\mathcal{B}(t, T, \mathbb{T})$, such that,

$$\begin{aligned} & \text{GB-VI}^{\text{bp}}(t, T, \mathbb{T}) \\ &= \sqrt{\frac{1}{T-t} \frac{1}{P_t(T)} \mathbb{E}_t \left(e^{-\int_t^T r_\tau d\tau} \int_t^T \mathcal{B}^2(t, T, \mathbb{T}) \|v_\tau(T, \mathbb{T})\|^2 d\tau \right)} \end{aligned}$$

That is,

$$\text{GB-VI}^{\text{bp}}(t, T, \mathbb{T}) = \mathcal{B}(t, T, \mathbb{T}) \times \text{GB-VI}(t, T, \mathbb{T})$$

- Nothing new here really, it's simply a new representation of the BP implied volatility index
- We now build on $\mathcal{B}(t, T, \mathbb{T})$ to construct a model-free measure of yield volatility based on both the BP and the percentage implied vols, $\text{GB-VI}^{\text{bp}}(t, T, \mathbb{T})$ and $\text{GB-VI}(t, T, \mathbb{T})$
- First, a few words on the interpretation of $\mathcal{B}(t, T, \mathbb{T})$

- We show that $\mathcal{B}(t, T, \mathbb{T})$ is an average of the squared forward,

$$\mathcal{B}(t, T, \mathbb{T}) = \sqrt{\int_t^T \omega_\tau \mathbb{E}_t^{Q_{v^\tau}} [F_\tau^2(T, \mathbb{T})] d\tau}, \quad \text{with } \int_t^T \omega_\tau d\tau = 1,$$

where the “realized variance probability”, Q_{v^τ} , has Radon-Nikodym derivative,

$$\rho(\tau; T) = \frac{dQ_{v^\tau}}{dQ_{FT}} \Big|_{\mathcal{F}_\tau} = \frac{\|v_\tau(T, \mathbb{T})\|^2}{\mathbb{E}_t^{Q_{FT}} [\|v_\tau(T, \mathbb{T})\|^2]}$$

- It distorts the fwd prob by giving more weight to the paths of $F_\tau(T, \mathbb{T})$ that have higher chances of experiencing episodes of high volatility

Back to BP yield volatility

Duration based

- Define

$$y_{\mathcal{B}}(t, T, \mathbb{T}) : \mathcal{B}(t, T, \mathbb{T}) = \frac{\text{GB-VI}^{\text{bp}}(t, T, \mathbb{T})}{\text{GB-VI}(t, T, \mathbb{T})} = \hat{P}(y_{\mathcal{B}}(t, T, \mathbb{T})),$$

and $D_{\mathcal{B}}(t, T, \mathbb{T})$, the modified duration of the guaranteed price $\mathcal{B}(t, T, \mathbb{T})$,

$$D_{\mathcal{B}}(t, T, \mathbb{T}) \equiv \frac{1}{1 + \frac{y_{\mathcal{B}}(t, T, \mathbb{T})}{n}} \left(\sum_{i=1}^N \omega_i \frac{i}{n} + \hat{\omega}_N \frac{N}{n} \right)$$

$$\omega_i \equiv \frac{\frac{C_i}{n} / \left(1 + \frac{y_{\mathcal{B}}(t, T, \mathbb{T})}{n}\right)^i}{\mathcal{B}(t, T, \mathbb{T})}, \quad \hat{\omega}_N \equiv \frac{100 / \left(1 + \frac{y_{\mathcal{B}}(t, T, \mathbb{T})}{n}\right)^N}{\mathcal{B}(t, T, \mathbb{T})}$$

- Model-free measure of duration-based yield volatility is,

$$\text{GB-VI}_{Y_d}^{\text{bp}}(t, T, \mathbb{T}) = 100 \times \frac{\text{GB-VI}(t, T, \mathbb{T})}{D_{\mathcal{B}}(t, T, \mathbb{T})},$$

or, using the definitions of \hat{P} and $D_{\mathcal{B}}$,

$$\begin{aligned} & \text{GB-VI}_{Y_d}^{\text{bp}}(t, T, \mathbb{T}) \\ & 100 \times \left(1 + \frac{1}{n} \hat{P}^{-1} \left[\frac{\text{GB-VI}^{\text{bp}}(t, T, \mathbb{T})}{\text{GB-VI}(t, T, \mathbb{T})} \right] \right) \times \text{GB-VI}^{\text{bp}}(t, T, \mathbb{T}) \\ = & \frac{100 \times \left(1 + \frac{1}{n} \hat{P}^{-1} \left[\frac{\text{GB-VI}^{\text{bp}}(t, T, \mathbb{T})}{\text{GB-VI}(t, T, \mathbb{T})} \right] \right) \times \text{GB-VI}^{\text{bp}}(t, T, \mathbb{T})}{\sum_{i=1}^N \frac{C_i}{n} \left(1 + \frac{1}{n} \hat{P}^{-1} \left[\frac{\text{GB-VI}^{\text{bp}}(t, T, \mathbb{T})}{\text{GB-VI}(t, T, \mathbb{T})} \right] \right)^{-i} + 100 \left(1 + \frac{1}{n} \hat{P}^{-1} \left[\frac{\text{GB-VI}^{\text{bp}}(t, T, \mathbb{T})}{\text{GB-VI}(t, T, \mathbb{T})} \right] \right)^{-N} \frac{N}{n}} \end{aligned}$$

where \hat{P}^{-1} denotes the inverse function of \hat{P}

- We also extend this gauge to the “post-issuance case,” where the maturity T of the forward is higher than the date of issuance of the bond

American corrections

- Some data might rely on American not European options
- Design algorithms to convert American vols into Europeans
 - Resulting index calculation is model-dependent
- Main idea—three steps
 - (i) Calibrate a pricing kernel to the *market* price of American options on futures
 - (ii) Use the American-implied pricing kernel to derive implications on the (unobservable) price of European options on forwards
 - (iii) Feed vol indexes through *model*-based data obtained in step (ii)

Time-deposits

- Only focus on LIBOR variance contracts for reasons of space—Rates not prices
- **Notation** $l_t(\Delta)$ is the simply comp IR on a deposit from t to $t + \Delta$ —say referenced to LIBOR
- Fwd contract is one where the payoff at T is $100 \times (1 - l_T(\Delta)) - Z_t(T, T + \Delta)$, where the forward price, $Z_t(T, T + \Delta)$ at t is $Z_t(T, T + \Delta) = 100 \times (1 - f_t(T, T + \Delta))$, where $f_t(T, T + \Delta)$ is the forward LIBOR
- $Z_t(T, T + \Delta)$ is a martingale under Q_{FT} ,

$$\frac{dZ_\tau(T, T + \Delta)}{Z_\tau(T, T + \Delta)} = v_\tau^z(T, \Delta) dW_{FT}(\tau)$$

- Therefore,

$$\frac{df_{\tau}(T, T + \Delta)}{f_{\tau}(T, T + \Delta)} = v_{\tau}^f(T, \Delta) dW_{FT}(\tau)$$

$$v_{\tau}^f(T, \Delta) \equiv (1 - f_{\tau}^{-1}(T, T + \Delta)) v_{\tau}^z(T, \Delta)$$

- Consider the BP case. The BP LIBOR integrated variance is,

$$V_t^{f, \text{bp}}(T, \Delta) \equiv \int_t^T f_{\tau}^2(T, T + \Delta) \|v_{\tau}^f(T, \Delta)\|^2 d\tau$$

- The fair value of the time deposit *rate*-variance swaps at time t , is

$$\begin{aligned} & \mathbb{P}_f^{\text{bp}}(t, T, \Delta) \\ &= \frac{2}{P_t(T)} \left(\int_0^{f_t(T, T+\Delta)} \text{Put}_t^f(K_f, T, \Delta) dK_f + \int_{f_t(T, T+\Delta)}^{\infty} \text{Call}_t^f(K_f, T, \Delta) dK_f \right), \end{aligned}$$

where

$$\text{Put}_t^f(K_f, T, \Delta) = \frac{\text{Call}_t^{\tilde{z}}(100(1 - K_f), T, \Delta)}{100}, \quad \text{Call}_t^f(K_f, T, \Delta) = \frac{\text{Put}_t^{\tilde{z}}(100(1 - K_f), T, \Delta)}{100},$$

and $\text{Call}_t^{\tilde{z}}(\cdot, T, \Delta)$ and $\text{Put}_t^{\tilde{z}}(\cdot, T, \Delta)$ are the OTM options on the forward LIBOR price Z

- Index is,

$$\text{TD-VI}_f^{\text{bp}}(t, T, \Delta) \equiv 100^2 \times \sqrt{\frac{\mathbb{P}_f^{\text{bp}}(t, T, \Delta)}{T - t}}$$

Pitfalls: maturity mismatch

- Suppose the option maturity is, say, one month, and the underlying is a forward expiring in five years
- We show model-free indexes cannot exist in this case
 - Need to apply a model-dependent correction term
- Intuitively, let T be the maturity of the option and S be the maturity of the forward, with $T \leq S$
 - “Option spanning” operates under the T -fwd prob
 - Forward risk and, then, its volatility, are defined under the S -fwd prob
 - Unless $T = S$, we would price the fwd risk volatility with the “wrong” probability
 - reminiscent of convexity problems arising in fixed income security evaluation

- Illustrate the government bonds case only
- **Notation** $F_t(S, \mathbb{T})$ is the fwd price at t , for delivery at S , of the coupon bearing bond expiring at \mathbb{T} . It satisfies,

$$\frac{dF_\tau(S, \mathbb{T})}{F_\tau(S, \mathbb{T})} = v_\tau(S, \mathbb{T}) \cdot dW_{FS}(\tau), \quad \tau \in (t, S)$$

- Only mention the percentage integrated variance,

$$V_t(T, S, \mathbb{T}) \equiv \int_t^T \|v_\tau(S, \mathbb{T})\|^2 d\tau$$

- Fair value of the variance swap referenced to $V_t(T, S, \mathbb{T})$ is,

$$\mathbb{P}(t, T, S, \mathbb{T}) = \mathbb{E}_t^{Q_{FT}}(V_t(T, S, \mathbb{T}))$$

- By a standard argument, for $\tau \in (t, T)$,

$$\frac{dF_\tau(S, \mathbb{T})}{F_\tau(S, \mathbb{T})} = v_\tau(S, \mathbb{T})(v_\tau(S, \mathbb{T}) - v_\tau(T, \mathbb{T}))d\tau + v_\tau(S, \mathbb{T}) \cdot dW_{FT}(\tau),$$

where,

$$dW_{FT}(\tau) = dW_{FS}(\tau) - (v_\tau(S, \mathbb{T}) - v_\tau(T, \mathbb{T}))d\tau,$$

is a multidimensional BM under Q_{FT}

- Therefore, by Itô's lemma,

$$-\mathbb{E}_t^{Q_{FT}} \left(\ln \frac{F_T(S, \mathbb{T})}{F_t(S, \mathbb{T})} \right) = -\mathbb{E}_t^{Q_{FT}} (\tilde{\ell}(t, T, S, \mathbb{T})) + \frac{1}{2} \mathbb{P}(t, T, S, \mathbb{T})$$

$$\tilde{\ell}(t, T, S, \mathbb{T}) \equiv \int_t^T v_\tau(S, \mathbb{T}) (v_\tau(S, \mathbb{T}) - v_\tau(T, \mathbb{T})) d\tau$$

- On the other hand, a Carr-Madan expansion leaves

$$-\mathbb{E}_t^{Q_{FT}} \left(\ln \frac{F_T(S, \mathbb{T})}{F_t(S, \mathbb{T})} \right) = 1 - \mathbb{E}_t^{Q_{FT}} \left(e^{\tilde{\ell}(t, T, S, \mathbb{T})} \right) + \frac{1}{2} \mathbb{P}_{\text{vix}}(t, T, S, \mathbb{T})$$

- Summing up,

$$\mathbb{P}(t, T, S, \mathbb{T}) = \mathbb{P}_{\text{vix}}(t, T, S, \mathbb{T}) + 2 \left(1 - \mathbb{E}_t^{Q_{FT}} \left(e^{\tilde{\ell}(t, T, S, \mathbb{T})} - \tilde{\ell}(t, T, S, \mathbb{T}) \right) \right),$$

- Volatility index is,

$$\text{GB-VI}(t, T, \mathbb{T}) = \sqrt{\frac{1}{T-t} \mathbb{P}(t, T, S, \mathbb{T})}$$

It's model-dependent as $\mathbb{P}(t, T, S, \mathbb{T})$ is

4/4 Credit

Issues of methodology

- Introduce credit variance swaps on *loss-adjusted forward default swap spreads*
 - Model-free price, expressed in terms of traded CDS index option prices
 - Percentage & BP indexes
- Need concept of *survival contingent probability* to account for default risk which is absent from our previous derivations, or equity
 - Market numéraire: *defaultable annuity*

The underlying risk

- CDS index
 - Buyer pays periodic premium—the CDS index spread
 - Seller insures losses from defaults by any of the index's constituents during the term of the contract
 - If a constituent defaults, the defaulted obligor is removed from the index, and the index continues to be traded with a prorated notional amount
- Options on a CDS index are European-style, to buy (payers) or sell (receivers) protection at the strike spread upon option expiry

Assumptions

- Credit events may occur over a sequence of regular intervals (T_{i-1}, T_i) with length $\frac{1}{b}$, for $i = 1, \dots, bM$
 - M is the no of years the index runs, T_0 is the time of the index origination
 - * E.g., $b = 4 \longrightarrow$ quarterly intervals
- We assume that
 - (i) Loss-given-default, LGD, is constant
 - (ii) The short-term rate r_τ is a diffusion process
 - (iii) Default arrives as a Cox process with intensity λ adapted to r

CDS indexes

- n is the initial number of names in the index decided at time $t \equiv T_0$, each constituent has notional value $\frac{1}{n}$, the same LGD, and the same intensity λ
- The number of names survived up to T_i is,

$$\mathcal{S}(T_i) \equiv \sum_{j=1}^n \left(1 - \mathbb{I}_{\{\tau_j \leq T_i\}}\right),$$

where τ_j is the time at which name j defaults, and the outstanding notional is,

$$\mathcal{N}(\tau) = \frac{1}{n} \mathcal{S}(\tau), \quad \mathcal{N}(t) \equiv 1$$

- Loss & premiums

- Index *loss* at τ_j should obligor j default is $\text{LGD} \frac{1}{n} \mathbb{I}_{\{t \leq \tau_j \leq T_{bM}\}}$
- *Premium* at T_i is $\frac{1}{b} \overline{\text{CDX}}_t(M) \times \frac{1}{n} \mathcal{S}(T_i)$ —constant premium determined at t times the outstanding notional at time T_i , $\frac{1}{n} \mathcal{S}(T_i)$
- Value of protection leg minus premium leg is:

$$\text{DSX}_t = \text{LGD} \cdot v_{0t} - \frac{1}{b} \overline{\text{CDX}}_t(M) \cdot v_{1t},$$

where

$$v_{0,t} \equiv \mathbb{E}_t \left[e^{-\int_t^{\tau_*} r(s) ds} \mathbb{I}_{\{t \leq \tau_* \leq T_{bM}\}} \right], \quad v_{1t} \equiv \sum_{i=1}^{bM} \mathbb{E}_t \left[e^{-\int_t^{T_i} r(\tau) d\tau} \cdot \mathbb{I}_{\{\text{Surv}_* \text{ at } T_i\}} \right],$$

and τ_* is the default time for an hypothetical representative firm with default intensity λ

- v_{0t} = value at t of \$1 paid off at the time of default of the repr firm, provided default occurs \leq index expiry
- v_{1t} = value at t of an annuity of \$1 paid at T_1, \dots, T_{bM} , until default of the representative firm or the expiry of the index, whichever occurs first
 - Or, value at t of a basket of defaultable bonds with zero recovery value issued by a representative obligor—a *defaultable annuity*

Forward starting indexes and credit default options

- A *forward starting index* is an index decided at t and running at T
- A *CDS index payer* \longrightarrow option to enter a CDS index as a protection buyer with strike spread K
- A forward starting index does not protect from any losses occurring \leq index begins
 - Upon exercise the protection buyer would also receive a *front-end protection* arising from losses occurring \leq index begins

- *Loss-adjusted forward default index* is,

$$\text{DSX}_{t,T}^L(\tau) = \frac{1}{b} \boxed{\mathcal{N}(\tau) v_{1\tau}} (\text{CDX}_{\tau}(M) - \overline{\text{CDX}}_t(M)),$$

where $\text{CDX}_{\tau}(M)$ is the value of $\overline{\text{CDX}}_{\tau}(M)$: newly issued forwards are worthless, viz $\text{DSX}_{\tau,T}^L(\tau) = 0$,

$$\frac{1}{b} \text{CDX}_{\tau}(M) = \text{LGD} \frac{v_{0,\tau}}{v_{1\tau}} + \frac{v_{\tau}^F}{\mathcal{N}(\tau) v_{1\tau}}$$

- Note, $\mathcal{N}(\tau) v_{1\tau}$ is the natural numéraire in this market

- Indeed, $CDX_t(M)$ is a martingale under the “survival contingent probability” Q_{sc} defined through

$$\left. \frac{dQ_{sc}}{dQ} \right|_{\mathbb{F}_T^r} = e^{-\int_{\tau}^T r(u) du} \frac{\mathcal{N}(T) v_{1T}}{\mathcal{N}(\tau) v_{1\tau}},$$

where \mathbb{F}_T^r denotes the information set at time T , which includes the path of the short-term rate only.

- Prices of a payer and receiver with strike K expiring at T , are, for any $\tau \in [t, T]$,

$$SW_{\tau}^p(K, T; M) \equiv \mathcal{N}(\tau) v_{1\tau} \cdot \mathbb{E}_{\tau}^{sc} \left[(CDX_T(M) - K)^+ \right],$$

$$SW_{\tau}^r(K, T; M) \equiv \mathcal{N}(\tau) v_{1\tau} \cdot \mathbb{E}_{\tau}^{sc} \left[(K - CDX_T(M))^+ \right],$$

Credit variance contracts

- Assume that

$$\begin{aligned} \frac{d\text{CDX}_\tau(M)}{\text{CDX}_\tau(M)} = & - \left(\mathbb{E}_\tau^{\text{sc}} \left(e^{j(\tau;M)} - 1 \right) \eta(\tau) \right) d\tau \\ & + \sigma(\tau; M) \cdot dW^{\text{sc}}(\tau) + \left(e^{j(\tau;M)} - 1 \right) dJ^{\text{sc}}(\tau), \end{aligned}$$

- *Percentage variance,*

$$V_M(t, T) \equiv \int_t^T \|\sigma(\tau; M)\|^2 d\tau + \int_t^T j^2(\tau; M) dJ^{\text{sc}}(\tau)$$

- *Basis point variance,*

$$V_M^{\text{bp}}(t, T) \equiv \int_t^T \text{CDX}_\tau^2(M) \|\sigma(\tau; M)\|^2 d\tau \\ + \int_t^T \text{CDX}_\tau^2(M) \left(e^{j(\tau; M)} - 1 \right)^2 dJ^{\text{sc}}(\tau)$$

- Only provide the pricing of *Basis point variance* in this presentation for reasons of space
- Moreover, we only consider “standardized contracts”—Chapter 5 in the book considers three contracts just as Part II of this presentation

- **Standardized BP-Credit Variance Swap Rate** The Standardized BP-Credit Variance Swap rate is the fixed variance swap rate $\mathbb{P}_{\text{var},M}^{*\text{bp}}(t, T)$, which zeroes the current value of

$$[V_M^{\text{bp}}(t, T) - \mathbb{P}_{\text{var},M}^{*\text{bp}}(t, T)] \times \mathcal{N}(T) v_{1T}$$

- We have,

$$\begin{aligned} & \mathbb{P}_{\text{var},M}^{*\text{bp}}(t, T) \\ &= \frac{2}{v_{1t}} \left[\sum_{i:K_i < \text{CDX}_t(M)} \text{SW}_t^r(K_i, T; M) \Delta K_i + \sum_{i:K_i \geq \text{CDX}_t(M)} \text{SW}_t^p(K_i, T; M) \Delta K_i \right] \end{aligned}$$

- Marking-to-market & replication issues in the book

Credit Volatility Index

- The *BP* Credit volatility index is,

$$\text{C-VI}_M^{\text{bp}}(t, T) \equiv 100 \times 100 \times \sqrt{\frac{1}{T-t} \mathbb{P}_{\text{var}, M}^{*\text{bp}}(t, T)}$$