Portfolio Optimization Using the Distribution Builder

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Intertemporal Consumption & Incomplete Markets

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Portfolio Optimization

- Classical Portfolio Optimization comes in two flavors
  - Universal Principles for optimization, e.g., Kelly criterion
  - Taking the preferences (or risk aversion) of the agent into account:
    - Classical: The preferences are given by a utility functions
    - Goes Back to D. Bernoulli (1738), axiomatization by Von Neumann–Morgenstern, long strain of financial mathematics literature starting with Merton

Bernoulli

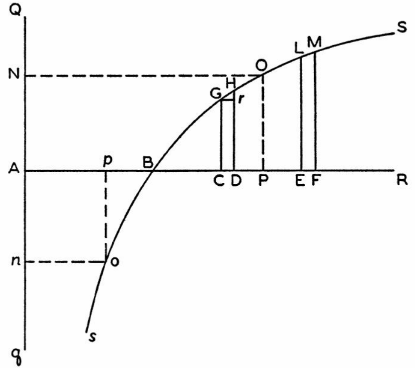


Figure: Bernoulli's original utility function

Portfolio Optimization

- Criticism:
  - Rational utility functions do not take into account the actual preferences of people (Kahnemann–Tversky, Quiggin,...)
  - Utility functions are not convex, rare extreme events are not adequately considered (Choquet integrals)
  - Should utility functions be descriptive or prescriptive?

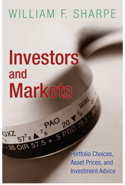
Portfolio Optimization

- Criticism:
  - Utility functions (or risk aversion) is very hard to measure for practical purposes
  - How to estimate? Are estimates consistent?

Distribution Builder

- Investors are notoriously bad in estimating their utility function
- Try instead to get more direct information from the agent
- Specifically, for terminal time portfolio optimization let the agent directly choose the desired distribution of terminal wealth that is reachable with given initial capital
- Distribution builder approach (Goldstein, Sharpe & Blythe; Goldstein Johnson and Sharpe; Monin)

Distribution Builder



future consumption. In a complete market setting such an investor's decision process can be summarized as follows:

$$\text{Budget} + \text{Prices} + \text{Preferences} \rightarrow \text{Distribution}$$

Given a budget, a set of state prices, and his or her preferences, the investor will choose the most desirable distribution—formally, the one that maximizes his or her expected utility.

Assume that an investor has chosen a distribution and that an outsider can observe the budget, state prices, and the selected distribution. From this information it may be possible to infer the investor's preferences:

$$\text{Budget} + \text{Prices} + \text{Distribution} \rightarrow \text{Preferences}$$

Figure: Philosophy of the Distribution Builder (Source: Sharpe)

Distribution Builder

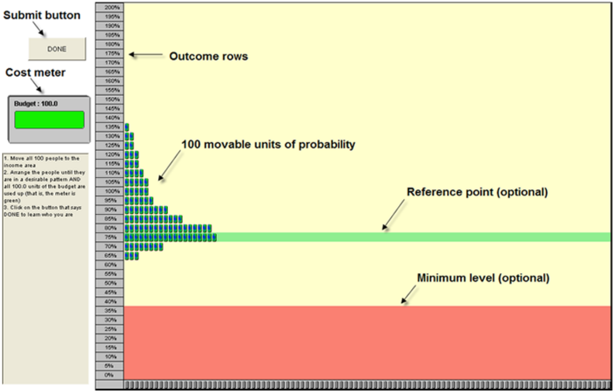


Figure: An implementation of the Distribution Builder (Source: Sharpe)

The cost efficiency principle

Idea

- We have to find the cheapest hedging strategy that produces at terminal time the required distribution.
- Thus, for given F, in a **complete** market, we try to solve

$$\inf_{X\sim F} \mathbb{E}\big[\xi X\big] = \inf_{X\sim F} c(X)$$

for pricing kernel ξ and replication pricing functional $c(\cdot)$

- Moreover, if ξ has a continuous distribution,

$$X^* = \arg\min_{X\sim F} \mathbb{E}\big[\xi X\big] = F^{-1}\big(1 - F_\xi(\xi)\big)$$

by the Fréchet-Hoeffding bounds

Rationalizingn of Investor's Behavior

- In a **complete market** we can always find a utility function U such that X^* is the optimal portfolio for expected utility maximization under U:
- We have by cost efficiency and duality

$$F^{-1}\big(1 - F_\xi(\xi)\big) = X^* = (U')^{-1}(\lambda\xi)$$

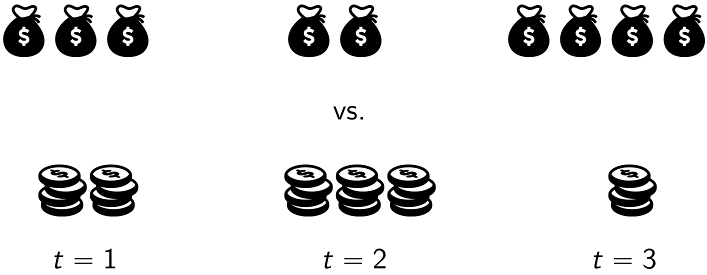
- Thus for $U(x) = \int_c^x F_\xi^{-1}\big(1 - F(y)\big)\,dy$ the portfolio X^* is **optimal** (Bernard, Chen, Vanduffel)

Consumption

- Can this approach generalized to incorporate consumption?
- Naive approach: Given desired a desired marginal distributions of consumption stream C_1, C_2, $\ldots C_n$, can we determine the price and hedge necessary
- Answer: Yes, just iterate the terminal market approach.

Consumption

Downside: All random variables are monotone functions of the pricing kernel and thus **serially correlated**



Consumption

- This might be not what an agent ones (who might hope that under-performance at one time is offset the next).
- More sophisticated approach: Let agent choose the joint distribution (or the copula additionally to the marginals)
- Existence of random variables corresponding to the joint distribution can be guaranteed using distributional transform (à la Rüschendorf) for discrete consumption
- Problem for continuous consumption stream so far unsolved

Consumption

- Practical implementation: use Gaussian or **Clayton copula**

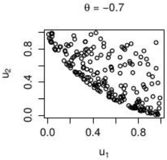


Figure: Bivariate Clayton Copula (Source: Ruppert)

- Generate distribution sample, calculate sum; generate pricing kernel sample; order them antimonotonically

$$
\begin{array}{cccc|c||c}
c_{11} & c_{21} & \cdots & c_{d1} & \sum_{i=1}^{d} c_{i1} & \xi_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_{1n} & c_{2n} & \cdots & c_{dn} & \sum_{i=1}^{d} c_{in} & \xi_n \\
 & & & & \uparrow & \downarrow
\end{array}
$$

Incomplete market

- One might ask how dependent these results are on the assumption of a complete market
- To do so, one has to establish a cost efficiency principle for incomplete markets
- The naïve guess would be that the optimizer X^* is a solution to

$$\inf_{X\sim F} \sup_{\xi\in\Xi} \mathbb{E}\left[\xi X\right] = \inf_{X\sim F} c(X)$$

  where the cost is given by the superhedging price $c(X) = \sup_{\xi\in\Xi} \mathbb{E}\left[\xi X\right]$

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- **That's wrong!**

Incomplete market

- We have to incorporate **superhedging**, over all pricing kernels $\xi \in \Xi$
- But how?

$$\inf_{X \sim F} \sup_{\xi \in \Xi} \mathbb{E}[\xi X]$$

or

$$\sup_{\xi \in \Xi} \inf_{X \sim F} \mathbb{E}[\xi X]$$

or are they even equivalent?

Incomplete Market

It turns out, the correct answer is

$$\sup_{\xi \in \Xi} \inf_{X \sim F} \mathbb{E}[\xi X] \tag{1}$$

Proposition

Assume that that the superhedging cost of some $X \sim F$ is finite. Then (1) has a unique solution (ξ^*, X^*) and if the optimal pricing kernel ξ^* has a continuous distribution, X^* can be expressed as

$$X^* = \left(F^{-1} \circ (1 - F_{\xi^*})\right)(\xi^*)$$

Incomplete Markets

- How about

$$\inf_{X\sim F} \sup_{\xi\in\Xi} \mathbb{E}[\xi X]?$$

- Does a minimax principle hold?

$$\sup_{\xi\in\Xi} \inf_{X\sim F} \mathbb{E}[\xi X] \stackrel{?}{=} \inf_{X\sim F} \sup_{\xi\in\Xi} \mathbb{E}[\xi X]$$

Incomplete Markets

- How about

$$\inf_{X \sim F} \color{red}{\sup_{\xi \in \Xi}} \mathbb{E}[\xi X]?$$

- Does a minimax principle hold?

$$\color{red}{\sup_{\xi \in \Xi}} \inf_{\color{blue}{X \sim F}} \mathbb{E}[\xi X] \stackrel{?}{=} \inf_{\color{blue}{X \sim F}} \color{red}{\sup_{\xi \in \Xi}} \mathbb{E}[\xi X]$$

this set is not convex

Convexification

Have to **convexify**: Denote the convex closure (wrt the topology of convergence in probability) of F-distributed random variables by

$$\overline{\mathrm{conv}}(F) := \overline{\mathrm{conv}\big(\{X \sim F\}\big)}^{L^0}$$

Proposition

We have that (Žitković; Bank & Kauppila)

$$\sup_{\xi \in \Xi} \inf_{X \sim F} \mathbb{E}\big[\xi X\big] = \sup_{\xi \in \Xi} \inf_{X \in \overline{\mathrm{conv}}(F)} \mathbb{E}\big[\xi X\big]$$
$$= \inf_{X \in \overline{\mathrm{conv}}(F)} \sup_{\xi \in \Xi} \mathbb{E}\big[\xi X\big] \leqslant \inf_{X \sim F} \sup_{\xi \in \Xi} \mathbb{E}\big[\xi X\big]$$

Convexification

Note: It turns out that we can say more if we assume F to be integrable: We have for distributions with support bounded from below

$$
\begin{gathered}
F \text{ is integrable} \\
\Longleftrightarrow \\
\overline{\mathrm{conv}}(F) \text{ is uniformly integrable} \\
\Longleftrightarrow \\
\overline{\mathrm{conv}}(F) \text{ is bounded in probability} \\
\Longleftrightarrow \\
\overline{\mathrm{conv}}(F) \text{ is convexly compact}
\end{gathered}
$$

Convexification

Proposition (He, Tang & Zhang)

Assume that the probability space is a standard Borel space. Let $X \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ and F be its cdf and assume that X is integrable. Without additional assumptions on the probability space,

$$\overline{\mathrm{conv}}(F) \subset \left\{ Y \in \mathcal{X} \,:\, Y \preceq_{cx} X \right\}$$

Furthermore if the probability space is atomless then

$$\overline{\mathrm{conv}}(F) = \left\{ Y \in \mathcal{X} \,:\, Y \preceq_{cx} X \right\}$$

(Here $Z \preceq_{cx} X^*$ means that Z is smaller than X^* in convex order, i.e., for all convex functions v $\mathbb{E}[v(Z)] \leqslant \mathbb{E}[v(X^*)]$.)

Incomplete market

Theorem

The price to superhedge an integrable cumulative distribution function F is given by

$$\inf_{X \leq_{cx} F} c(X). \tag{2}$$

The optimizer X^* provides **at least** *the same expected utility for* **any** *concave utility function.*

Notes

- In general we have a strict inequality,

$$\sup_{\xi\in\Xi}\inf_{X\sim F}\mathbb{E}\big[\xi X\big]\overset{(*)}{=}\sup_{\xi\in\Xi}\inf_{X\in\overline{\mathrm{conv}}(F)}\mathbb{E}\big[\xi X\big]$$

$$=\inf_{X\in\overline{\mathrm{conv}}(F)}\sup_{\xi\in\Xi}\mathbb{E}\big[\xi X\big]<\inf_{X\sim F}\sup_{\xi\in\Xi}\mathbb{E}\big[\xi X\big]$$

- Note that while (*) denotes an equality of the value functions, the optimizers do not agree in general.

Conclusions & Limitations

- We have proved a cost-efficiency principle for consumption in complete markets and for terminal wealth in incomplete markets
- With this, results from complete markets relying on cost efficiency can be carried over to incomplete markets. In particular, we can implement the distribution builder.
- Condition that optimizer ξ^* has a continuous distribution is hard to check in practice
    - If all pricing kernels are uniformly absolutely continuous
    - Use model specific tools to prove it in concrete examples

Thank you!

Thank you!

Example

(Counter-)example for minimax without convexification:

Mixture SV model:

$$S_t = s_0 e^{\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t}, \quad \sigma = \begin{cases} \sigma_H & \text{prob. } p \\ \sigma_L & \text{prob. } 1 - p \end{cases}$$

Pricing kernels:

$$\xi^q = \frac{q}{p}\mathcal{E}\left(-\int_0^{\cdot} \frac{\mu - r}{\sigma_H} dW_t\right)_T \mathbb{1}_{\{\sigma = \sigma_H\}} + \frac{1-q}{1-p}\mathcal{E}\left(-\int_0^{\cdot} \frac{\mu - r}{\sigma_L} dW_t\right)_T \mathbb{1}_{\{\sigma = \sigma_L\}}$$

Choose $F \sim S_T$

Example

- $\inf_{X \sim F} \sup_{\xi \in \Xi} \mathbb{E}[\xi X]$ has $\xi^* = \xi^0$ or $\xi^* = \xi^1$
- However, for $\sup_{\xi \in \Xi} \inf_{X \sim F} \mathbb{E}[\xi Y]$ we have

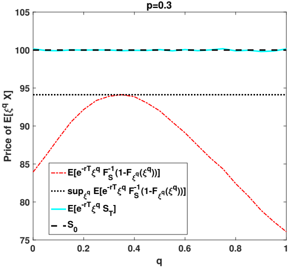

Figure: Counterexample