Liability Driven Investment: A Dynamic Hedging Strategy Against Multiple Risk Exposures of Pension Funds

Shaojie He, Jin Li, Sandeep Saju, Mengying Zhang, Yanan Zhang, Jiahe Zhou
Cornell MFE team, Advisor: Sasha Stoikov, Victoria Averbukh

Abstract
In this paper, we first discuss the background of pension crisis, demographic risk and several market risks of pension funds. We build a liability actuarial model to track the evolution of defined-benefit pension liability and the change of its present value. We construct a dynamic duration-matching strategy with proper futures contracts. We consider both the term structure change of interest rate and the credit spread between Treasury bonds and corporate bonds and find out that the combination of corporate bond and ultra long-term Treasury futures is the best strategy hedging these risks by back-test. Based on futures duration-matching strategy, we back test a different asset allocation, which consists of stock, Treasury bonds and corporate bonds. We also construct statistical indicators to identify the optimal portfolio. The results show that corporate bonds should be assigned a heavier weight within asset portfolio, and more weight should be assigned to stocks in the under-funded scenario.

Keywords
Liability-Driven Strategy, Duration-Matching, Dynamic Hedging

Contents

Introduction

"During the next decade, you will read a lot of news-bad news-about public pension plans. I hope my memo is helpful to you in understanding the necessity for prompt remedial action where problems exist."-Warren Buffet commented in his 2013's annual letter to Berkshire Hathaway shareholders. The corporate pension plans are not in a good shape either. "More than two-thirds of the companies that make up the S&P 500 have defined-benefit pension plans, and as of last quarter only 18 were fully funded." TIME magazine wrote in September 2012. The looming pension crisis in the OECD countries raises the need to reform the pension systems.

In this study, we analyze the underlying causes of the pension crisis and model the pension liabilities. Based on this model, we propose multiple hedging strategies to de-risk the pension plan. Back-testing is conducted to examine the results of each strategy.

This study is organized as follows. Section 1 reviews the background of pension crisis and pinpoints the major causes of the pension funding problem. Section 2 introduces our approach to model pension liabilities. Section 3 discusses our proposal to manage the pension risk which compares four different hedging strategies. In section 4 we examine the results of each investment strategy based on the back-tested data. Section 5 offers recommended approaches for reducing the pension gap.

1. Pension Crisis

1.1 Background

Many organizations offer pension plans to their employees. These pensions are paid through a pension fund which is managed independently from the sponsoring organization. The pension fund collects contributions from future pensioners (current employees) and from the sponsoring organization. These plans fall into two main categories - defined-contribution and defined-benefits pension plan. In a defined-benefit system, which we are going to focus on, a fixed retirement income is promised to their members. This guarantee introduces a considerable amount of risk to the pension fund. The aim of the pension fund will be to achieve sufficient performance from their assets to match the guaranteed pension benefits without additional contributions from the sponsoring organization.

1.2 Pension Fund Health Measurement

The health of a pension fund is determined by indicators called funding ratio and funding gap. The present value of projected pension benefits is treated as the liabilities. The funding ratio of a pension fund at time t is defined as:

$$F_t = Funding\ ratio = \frac{Asset\ value}{Liabilities} = \frac{A_t}{L_t} \tag{1}$$

Where A_t represents the asset value of the fund (i.e. the result of the portfolio strategy, net of past payments and contributions), and L_t represents the present value of the liabilities.

When a pension fund's liabilities become greater than its assets, the pension fund is underfunded and is considered risky. This is because pension fund will find it difficult to meet its pension obligations with the limited assets set aside to fund them. As we saw earlier, a huge proportion of corporate, state and federal pension funds in OECD countries are underfunded resulting in a looming pension crisis.

1.3 Decomposing Risks

There are several factors that lead to this widespread under-funding of pension funds.

- **Shifting Demographics**
  Increasing life expectancy coupled with a fixed retirement age and a decrease in the fertility rate in the majority of the developed world lead to lower support ratio (ratio of workers contributing to the pension to pensioners receiving the pension benefits). This results in a lower growth of assets compared to faster growth of the liabilities.
- **Volatile Market Returns & Low Interest Rates**
  Since the liabilities are computed by calculating the present value of the projected pension benefits, lower interest rate increases the liabilities. Since financial crisis, market returns on the asset investment has not generated enough to compensate for the increase in liabilities due to lower interest rates.
- **Salary Inflation**
  Since it's common for defined-benefit pension funds to base their payments to beneficiaries on average salary of the final years, there are salary inflation risks. The salary inflation risks could be attributed to total inflation of the society or to the increase of the real term salary in the industry, because of the booming economy. Thus there's a theory saying that we should invest in stocks, since stock prices also go up in the booming economy (Fischer Black, 1989).

We need to find a solution to hedge these risks. Typically, when we think of reducing risk, we think of decreasing the volatility of a portfolio, but pension risk focuses on asset volatility in relation to the liability. Therefore, the best way to reduce pension risk is to hold an asset or a portfolio of assets that behaves like the liability.

To reduce the volatility of the portfolio in relation to the liability, we need to link pension liabilities to assets, in which decomposing the risk factors in the liabilities is a crucial part (Meder, Aaron, and Renato Staub, 2007). Besides decomposing the risk factors in liabilities, the liability-driven-investment (LDI) solutions can be adopted to manage the pension plan. The LDI solutions treat the liability stream as the plan's benchmark and consequently consider all risk and return characteristics relative to the liability stream. In this context, the aggregate investment portfolio is perceived as a hedging portfolio and a return-generating portfolio. Kurt Winkelmann et al. (2007) separated the hedging issue into two separate portfolios with the first one mimicking the risk characteristics of the liabilities and the second one being the actual hedging portfolio whose design is based on the liability benchmark. Cash bonds, futures or interest rate swaps were suggested to be included in the portfolio to hedge the liabilities. They suggested that the choice of hedging vehicle can be framed as how much basis risk (tracking error between the liabilities and the hedging portfolio) the investor is willing to tolerate.

However, in our study, we are going to focus more on hedging the market risks of the pension liability. We are going to assume that the liability could be projected, and we are not going to consider the salary inflation risk.

2. Liability Model

2.1 Motivation

Since most pension funds keep their data confidential, we build a liability model based on actuarial mathematics to simulate the evolution of cash flow and duration of liability through the passage of time. This model is going to be used to back test our hedging strategies from late 2006 to the end of 2014.

2.2 Assumptions

The assumptions we make are as follows,

- In this pension plan, cash flows are static, which means that throughout years, no new members join the plan, and the number of future pension payments is decaying.
- The initial age distribution of the participants in this pension plan follows the US working force population density in 2005.
- The participants' age ranges from 20 to 100, and we take the cash outflow in the future 100 years into consideration.
- The force of mortality follows Makeham's Law with standard-valued parameters.
- The pension is quarterly payment of $1000, with no adjustment for inflation or CPI.
- The retirement age of 65 is constant over time, which means employees start to get pension payments after the age of 65.

2.3 Data

Two data sources are used as follows,

- Citigroup Pension Discount Curve and Liability Index, with monthly data from 30/9/2006 to 12/31/2014.
- U.S. Bureau of Labor Statistics, Civilian Labor Force and Participation Rates With Projections, with data in 2005.

2.4 Model

Let (x) denote a life aged x with $x \geq 0$, $S_x(t)$ denote the probability that (x) survives for at least t years.

In actuarial mathematics, the force of mortality represents the instantaneous rate of mortality at a certain age measured on an annualized basis. We denote the force of mortality at age x by μ_x.

With Makeham's Law, we have

$$\mu_x = A + Bc^x \tag{2}$$

From actuarial mathematics, we derive that

$$S_x(t) = s^t g^{c^x(c^t-1)} \tag{3}$$

where $s = e^{-A}$ and $g = exp\{-\frac{B}{logc}\}$, and here we take the standard values $A = 0.00022$, $B = 2.7 \times 10^{-6}$ and $c = 1.124$. [See Appendix]

In order for a person currently aged x to receive a payment of $1000 at future time t, there are two conditions to meet:

1. This person is still alive at future time t
2. This person reaches the retirement age at future time t, so that he is eligible to receive pension payment

Therefore, according to condition (2), any combination of (x,t) such that $x+t \leq 65$ will ensure that (x) gets no pension payment at future time t.

Based on conditions (1) and (2), a person aged x with $x+t > 65$ will receive a $1000 pension payment only if he is still alive at time t. Thus the expected pension payment is $\$1000 * S_x(t)$.

In conclusion, the pension payment made to a person currently aged x at future time t is given by,

$$P(x,t) = \begin{cases} 0 & \text{if } x+t \leq 65 \\ \$1000 * S_x(t) & \text{if } x+t > 65 \end{cases} \tag{4}$$

2.5 Procedure

We follow the following procedure to build the liability model,

1. From equations (3) and (4), we get a 81×400 matrix showing all the $P(x,t)$ with x ranging from 20 to 100, t ranging from 0.25 to 100.
2. For a given future time t_i, the total cash outflow V_{t_i} is the weighted sum of all $P(x,t_i)$ with x ranging from 20 to 80, weights given by the corresponding proportion of people aged x in the entire work force.
   $V_{t_i} = \sum_{j=20}^{80} \omega(x_j) * P(x_j, t_i)$.
   According to the raw data, the work force population is shown in Figure 1.

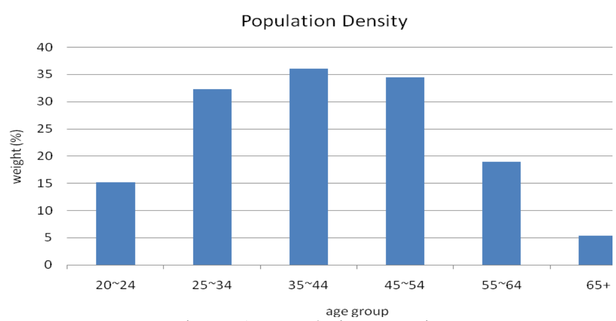

Figure 1. Population Density

   In order to assign specific weights to each age within the age groups, and to get a smooth curve showing the work force population density, we use exponential decay or inverse exponential decay and some other mild adjustments to obtain the adjusted population density shown in Figure 2.

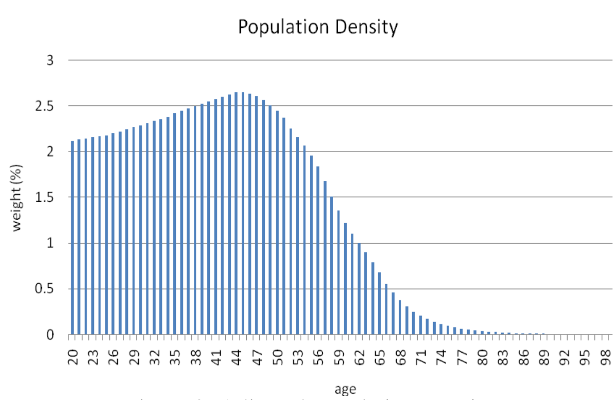

Figure 2. Adjusted Population Density

   Using these weights, we get the weighted-summed cash flows for t from 0.25 to 100.
3. As time goes by, the outstanding obligation $\overrightarrow{\Pi}_t$ (pre-discounted) will be reduced by the nearest due cash outflow after the quarterly payment date. That is to say,
   $\overrightarrow{\Pi}_{t_{i+0.25}} = (V_{t_{i+0.25}}, V_{t_{i+0.5}}, \ldots, V_{t_{100}})$,
   while $\overrightarrow{\Pi}_{t_i} = (V_{t_i}, V_{t_{i+0.25}}, \ldots, V_{t_{100}})$.
4. Using Citigroup Pension Discount Curve and Liability Index, we calculate the present values of $\overrightarrow{\Pi}_{t_i}$ at time t_i. However, the discount rate is computed semi-annually, and we need to evaluate it quarterly. We used the average of two consecutive values of discount rate to approximate the missing discount rate values. Also, since we need discount rates for cash flow present values' calculation for the years longer than 30, we assume that they all equal to the discount rate in year 30. Then we get the discount rate curves in Figure 3.

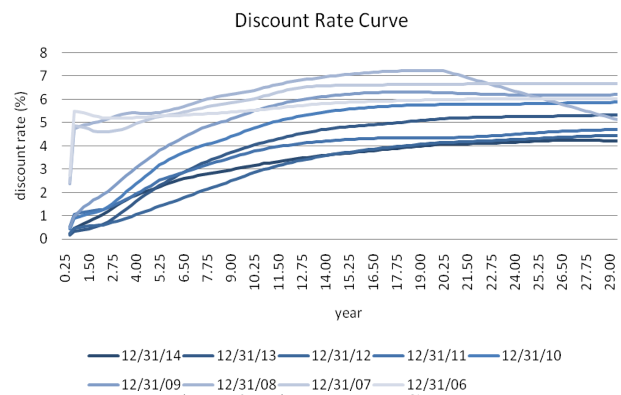

Figure 3. Discount Rate Curve

   Figure 4 shows evolution of net present value.

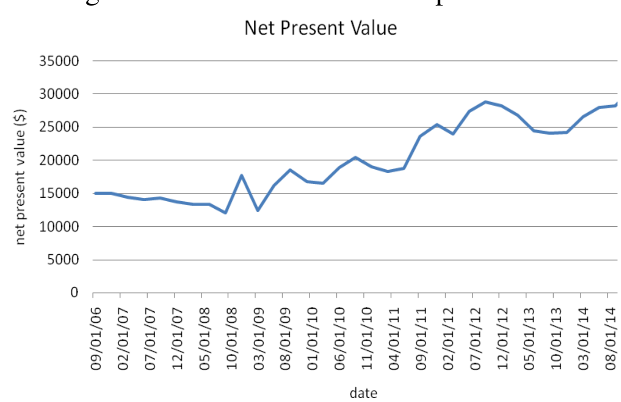

Figure 4. Net Present Value

5. Calculate the Macaulay duration at each time $t_0 = 09/30/2006$ to $t_{34} = 12/31/2014$ by computing the weighted average of the term of each cash flow.
6. Calculate the modified duration by dividing the Macaulay duration by $(1 + r_t)$, with r_t given by Citigroup Pension Discount Curve and Liability Index.

2.6 Result

We get Figure 5 and 6 for cash flow at t_0 and duration respectively.

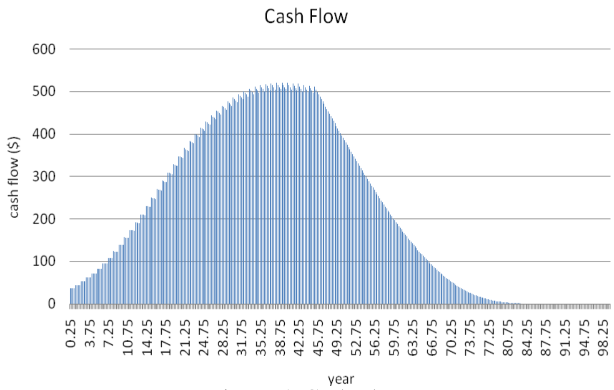

Figure 5. Cash Flow

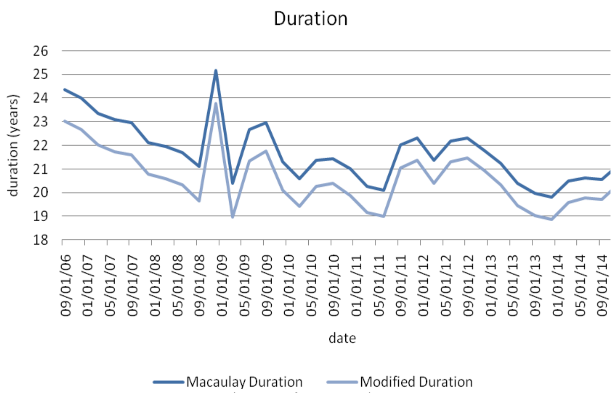

Figure 6. Duration

3. Hedging Strategies

To hedge the risks on the pension side, we have developed four hedging strategies, ranging from the most basic to more complicated as follows.

3.1 Vanilla Strategy

One straightforward intuition of hedging is to use fixed income products, because the biggest risk involved is interest rate risk. So our vanilla hedging strategy is to invest all assets in Treasury bonds. Here we use iShares Barclays 20+ Yr Treas. Bond ETF (TLT) as our investment instrument.

The calculation process is simple. We just match the value of asset and liability and then track the changes of the funding ratio from 2006 to 2014.

3.2 Duration-Matching Strategy

Because the duration of TLT is approximately between 13 and 18 years, which is shorter than the duration of the pension liability, we would like to invest in instruments with longer duration to match our liability, in order to make our portfolio less sensitive to interest rate changes. Therefore, we add Treasury futures to our portfolio. Besides TLT, we also invest in 10-Year US Treasury Note futures, which has the highest trading volume within the bond futures asset class.

Allocation ratio is determined by duration matching condition. The calculation formulas are as follows,

$$A_t \cdot D_t^a \cdot \Delta y_t + N_t^f \cdot P_t \cdot M \cdot D_t^e \cdot \Delta y_t = L_t \cdot D_t^l \cdot \Delta y_t \tag{5}$$

$$A_t + N_t^f \cdot M \cdot MR = TA_t \tag{6}$$

where A_t = Asset, D^a = Modified Duration of Asset, Δy = Change in Yield, N^f = Number of Futures Contracts, P = Quote Price of Futures, M = Multiplier (Futures Contract Size), L = Pension Liability, D^l = Modified Duration of Liability, MR = Margin Rate, TA = Total Asset, D^e = Empirical Duration, which is calculated with historical data by linear regression model as follows,

$$\frac{\Delta P_t}{P_t} = \alpha + \beta \Delta y_t + \varepsilon_t \tag{7}$$

where α, β and ε are regression parameters, β is the empirical duration, $\frac{\Delta P_t}{P_t}$ is the percentage change in price at time t, and Δy_t is the change of yield at time t. Note that the total asset at the very first period equals liability, but does not equal to each other in the following periods.

3.3 Modified Duration-Matching Strategy

Despite we can obtain a duration matching at the beginning with a portfolio of TLT ETF and 10-Year US Treasury futures, it does not hedge well in the years after. There are mainly two reasons that can explain this phenomenon.

- The non-parallel moves in the yield curves of bonds with different maturities. To be more precise, because the liability generally has an expected maturity of 100 years, but both the Treasury ETF and the Treasury futures we invest in have much shorter maturity, it is quite unlikely that their yield curves move in the same way.
- The corporate spread is ignored here. The cash flows from the liability side is discounted with a rate which has taken corporate bond yield into consideration, which means that the pension liability has corporate spread exposure. Only investing in Treasury and Treasury futures cannot hedge corporate spread risk.

Due to the above reasons, we modify our duration-matching strategy by investing in corporate bond ETF and futures on bonds with longer maturities. To be more specific, the corporate ETF we use here is iShares iBoxx $ Investment Grade Corporate Bond ETF (LQD) and iShares 10+ Year Credit Bond ETF (CLY), and the longer maturity futures we use is Ultra US Treasury Bond Futures. The reason why we use two ETF rather than one is because of our preference to longer duration and more historical data. LQD has longer history and CLY has longer duration. Thus from 2006 to 2009, we invest in LQD and from 2009 to 2014, we invest in CLY. Again, the allocation ratio is determined by duration-matching condition, which is similar to the calculation process of duration-matching strategy. The only one difference is that we divide the yield change into two parts, the change in risk free rate and the change in credit spread. Therefore, in addition to formulas (5), (6) and (7), we add,

$$\Delta y = \Delta r + \Delta s \tag{8}$$

where Δy = Change in Yield, Δr = Change in Risk Free Rate, Δs = Change in Credit Spread. Here we choose the number of futures N_f that match the coefficients of Δr on both sides.

3.4 Equity Strategy

According to our research, in reality, many pension funds invest in other asset groups such as equities, commodities and currencies, etc., besides bonds and fixed income derivatives, in order to get high growth or compensation for their transaction and administrative costs. Therefore, we further diversify our portfolio by investing in stocks as well. We invest in iShares Core S&P Total US Stock Market ETF (ITOT) here. The asset allocation ratio is determined by optimization techniques to get the most stable funding ratio throughout our calculation period.

4. Back Testing

4.1 Vanilla Strategy

Although it is common for a pension fund to hold stocks, holding bonds could be a better strategy in the sense of liability driven investments. The exposure to interest rate through holding a bond parallels with liability's exposure to interest rate change.

In Figure 7, we back test the strategy that is fully invested in Treasury bond or in stocks. Through the back testing processes, we extract a certain amount of money from the pension fund asset seasonally to pay for the beneficiaries and compare the remaining asset with the remaining present value of liability. We can see that the Treasury bonds matched with liabilities far better than stock investment strategy. We call the strategy that invests all the asset into Treasury bond the Vanilla Strategy, and we view the Vanilla Strategy as the benchmark strategy.

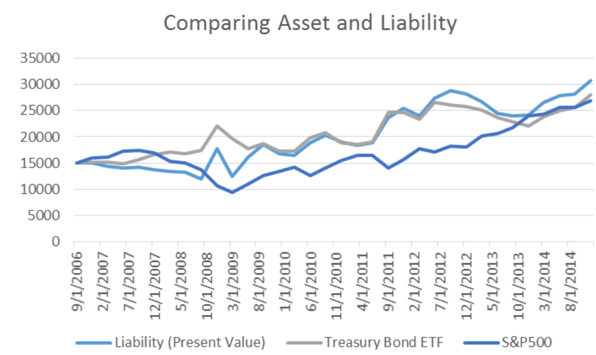

Figure 7. Comparing Asset and Liability

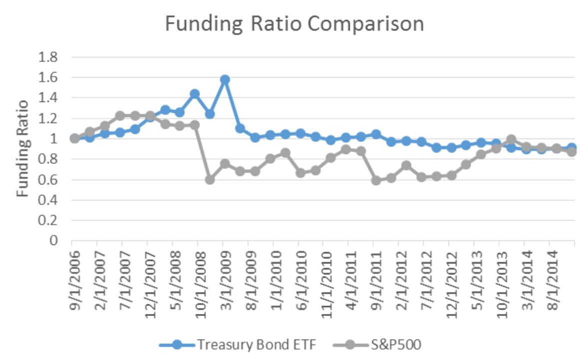

Figure 8. Funding Ratio Comparison

There are problems with the Vanilla Strategy, since the effective duration of iShares 20+ Year Treasury Bond ETF is roughly 17 which is shorter than the average 25-year duration of liability. We can see from Figure 7 that there is a widening gap during 2008 financial crisis and during 2012, which leads to a worsening funding ratio. This fact can partially be attributed to the mismatch of duration between liability and asset.

4.2 Duration-Matching Strategy

The main idea of the duration-matching strategy is to match the duration of liability and asset, balancing the level of interest rate risk exposure. We use long position in Treasury Futures to leverage our interest rate exposure and extend the duration of assets.

Choice of Futures Contract Since we use quarterly data in our back-test, we also re-balance our futures position on a quarterly basis. We trade the front futures contract, the most actively traded contract, and switch to the next contract one week before the delivery date.

With about 25-year duration on the liability side, we want to use the futures with the underlying asset of longest-term. At the same time, however, we also have to take the trading volumes into consideration which can influence the transaction liquidity and the scale of futures basis risk. Putting all concerns together, we use 10 Year Treasury Note Futures, which has a relatively long maturity and the biggest exchange volume in Treasury Futures. Therefore we first assume roughly parallel move of interest rate term structure and see the result of hedging with the most actively traded 10 Year Treasury Note Futures.

Strategy Performance and Spread Exposure We can see from Figure 9 that our futures hedging strategy improves the funding ratio and our funding ratio is no long below 100% as the Vanilla Strategy. The Figure 9 shows that the 10-year Treasury yield has a downside trend along our time horizon, which drives the present value of liability up. However, with the duration hedging strategy, we gain in the futures long position as well, filling the gap of asset-liability duration mismatch and rendering the funding ratio above 100%.

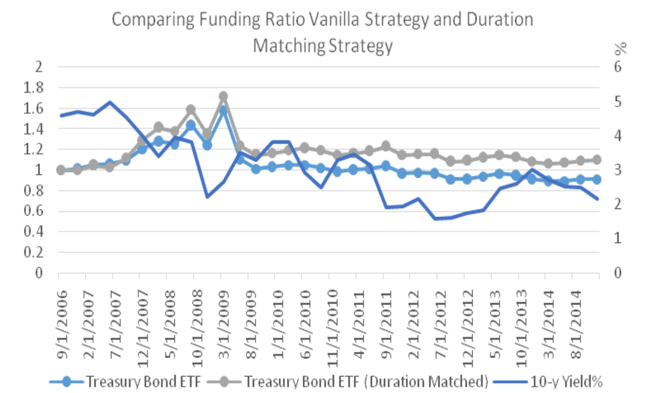

Figure 9. Comparing Funding Ratio of Vanilla Strategy and Duration Matching Strategy

In the duration-matching strategy, although the funding ratio fluctuates smoothly during most of the time, there is a great up-and-down around 2008 financial crisis. In this case, it hasn't brought much trouble to us because our funding ratio is always above 100%. However, as long as the total portfolio has remaining risk exposures, we may lose a lot when the market turns to the opposite side.

In order to find which exposure we still have in our total portfolio, we calculate the spread of 25-year AA Corporate Yield minus 10-year Treasury Yield. We can interpret this spread as the combination of credit spread (the spread between corporate yield and Treasury yield) and interest term structure spread (the spread between 10-year Treasury yield and 25-year Treasury yield).

In Figure 10, we can see that the spread moves almost identically with the funding ratio. Since the average duration of liability is about 25, it has the exposure both 25-year interest rate and the credit spread. While, on

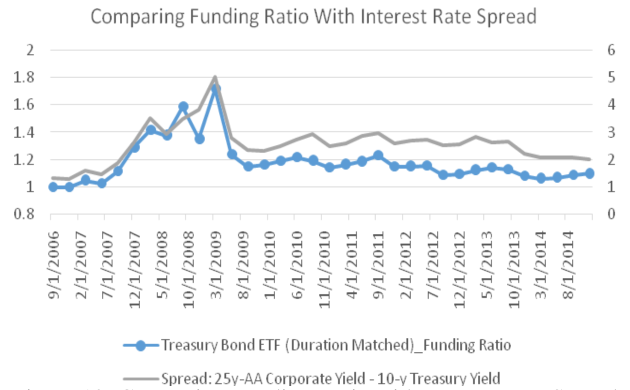

Figure 10. Comparing Funding Ratio with Interest Rate Spread

our asset side, the Treasury Bond ETF has exposure to about 17-year interest rate and 10 Year Treasury Futures only has exposure to 10-year interest rate. Thus the total portfolio has exposure both to term structure spread and credit spread. When the spread combination rises extremely during the 2008 financial crisis, the liability goes down a lot, comparing to the asset. Thus in the next section, we are going to modify our strategy and switch to different financial instruments in order to relieve those two spread exposures.

4.3 Modified Duration-Matching Strategy

Since we are going to hedge credit spread and term structure spread, we use corporate bond ETFs as the asset and start considering about Ultra Treasury Futures (with 25+ year underlying asset). Although the trading volume is not very high, it brings exposure to longer-term interest rate. We hold corporate bond in order to get credit spread exposure on the asset side, and we use Ultra Treasury Futures as the duration hedging instrument in order to extend exposure to longer term interest rate.

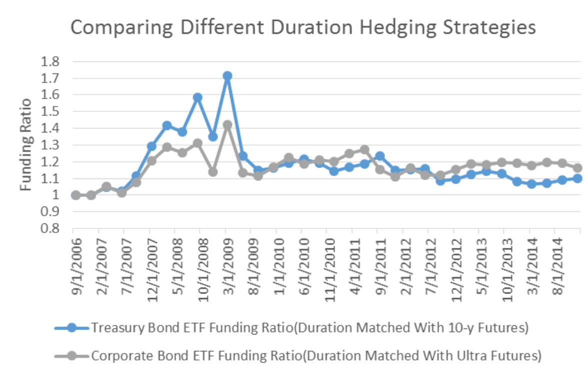

Figure 11. Comparing Different Duration Hedging Strategies

From Figure 11, we can see that the modified duration matching strategy with corporate bond and Ultra Treasury

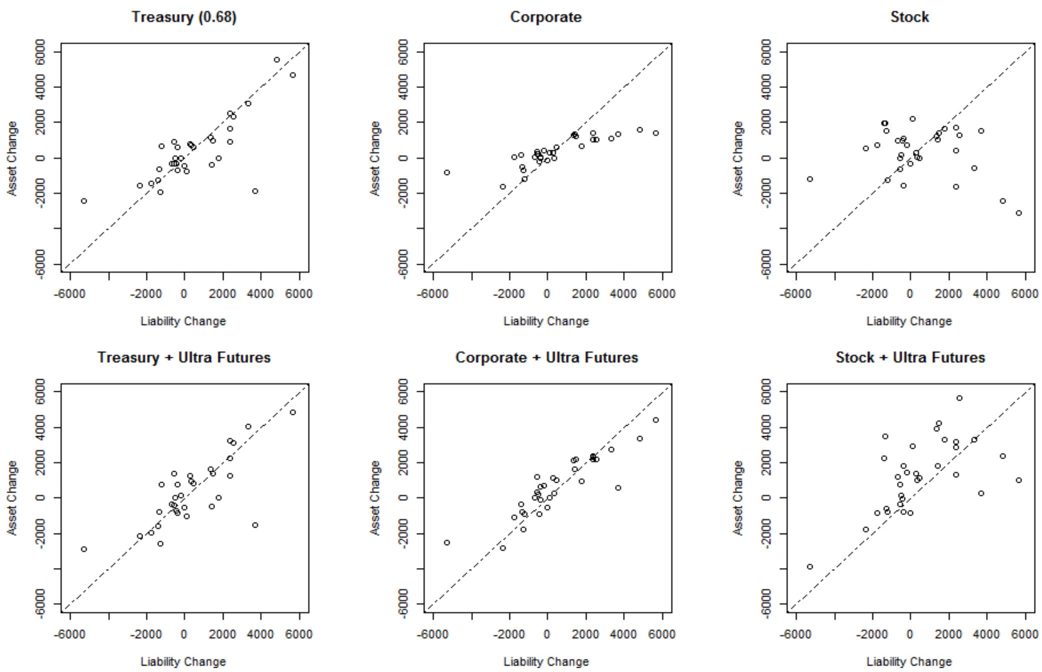

Figure 12. Scatter of Liability Change Against Asset Change

Futures performs better than the original duration, in terms of liability-driven strategy.

Modified duration-matching strategy relieves dramatic changes during the financial crisis. Moreover, the funding ratio is quite stable after 2012.

Figure 12 shows the performance of the liability-driven strategy. If asset and liability move identically, the scatter plot should follow the line with the slope of 1. Comparing strategies of holding corporate with and without futures, we could see that the futures position rotate the scatters towards the dot line. The same also happens to the strategy of holding stocks with and without futures.

4.4 Equity Strategy

In a more advanced strategy, we allow investment in the equities market. The initial funding ratio will influence our decision significantly. Under the fully-funded scenario, the major task for the pension fund is to maintain that trend. In the under-funded scenario, more excess return is desired to fill the funding gap. We employ two benchmarks to measure the performance of strategies and determine the optimal investment plan under different scenarios.

In addition to our investment in fixed income market, we also include some amount of equities within our portfolio in this part. S&P 500 is one of the most popular benchmarks in equity market, and we invest in iShares Core S&P Total U.S. Stock Market ETF which tracks the performance of S&P 500 index.

4.4.1 Fully-Funded Scenario

Variability-Reduction Method In order to measure whether the risk for a certain investment strategy is fully hedged, we introduce a risk measure called variability-reduction ratio. The formula is

$$VR = 1 - \frac{\sum_{i=1}^{n}(X_i - Y_i)^2}{\sum_{i=1}^{n} Y_i^2} \tag{9}$$

Where X_i is the quarterly value change of asset and Y_i is the quarterly value change of liability.

The intuition is that if all the risks of liability are perfectly hedged, then the daily value change of asset and liability should be offset by each other. In this case, the numerator $\sum_{i=1}^{n}(X_i - Y_i)^2$ should be very close to 0 and the VR ratio goes to 1. If the hedging strategy works poorly, the numerator will become very large because of the square operation, which leads to a significant drop of VR. If we start from a 100% funding ratio at the beginning, the goal of the manager is to seek for a high VR ratio.

Construction of Optimal Portfolio The basic idea is to choose the proportion of Treasury notes, corporate bonds, Treasury bond futures and S&P 500 to maximize VR ratio. We permits short selling of the ETF on each asset, but the short amount cannot exceed the total amount of asset in order to control the risk. The result is presented in Figure 13.

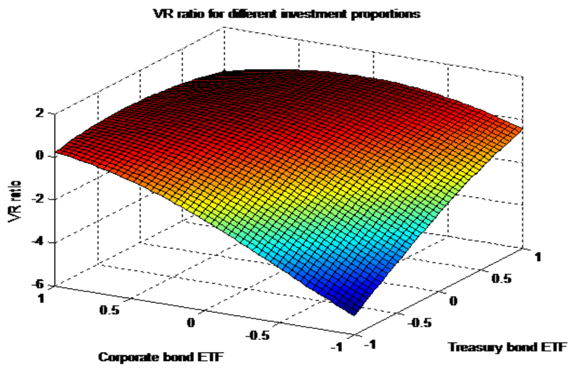

Figure 13. VR Ratio for Different Investment Proportions

The value of the VR ratio is presented based on the proportion of Treasury note ETF and corporate bond ETF. The rest of money is invested in the S&P 500 ETF. According to VR ratio surface, it is obvious that buying more bonds leads to a higher VR ratio. The largest VR ratio corresponds to the strategy that 92% going to corporate bond ETF, 12% to Treasury bond ETF and shorting 4% for S&P 500 ETF.

Surprisingly, we invest little in equity market in the optimal portfolio. It might sound inadvisable since most of the equities did an excellent job during the past years. The ETF that tracks S&P 500 tripled from late 2008 to now. However, the extraordinary growth rate is exactly the reason why we exclude it. If we start from 100% funding ratio, the ultimate goal for pension funds is to maintain stability. Some exposure to certain risks are essential if we want to increase the possibility to generate excess return, while these risk factors also lead to significant loss if market goes in unexpected direction. Suffering from under-funding problem is very dangerous since there is a potential that the fund fail to fulfill the obligations to beneficiaries. Actually, as we can notice from Figure 14, the growth in the equity market in past years was very lucky and unprecedented owing to the recovery from financial crisis and the quantitative easing (QE) program from Federal Reserve. In the past years, a slightly short position in equity market may help to offset the risk exposure in fixed income market as they were moving hand in hand during that period.

In conclusion, if our objective is to maintain a stable funding ratio around 100% and avoid any unfavorable risks, staying away equity market might be a better choice since the volatility there is much higher than fixed income

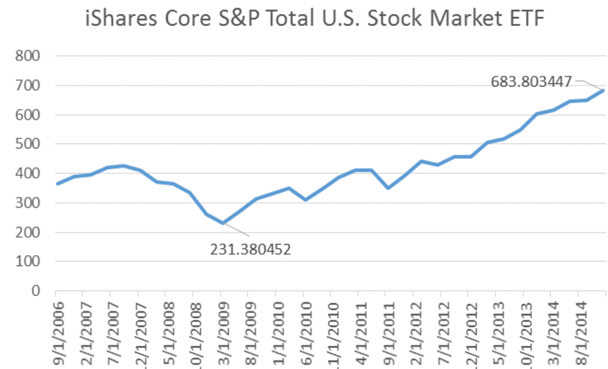

Figure 14. iShares Core S&P Total U.S. Stock Market ETF

market, and the low correlation between equities and our liabilities makes it hard to match their risk exposures. The optimal strategy we come up with focuses on corporate bonds which embrace both interest rate risk and credit risk. Since most corporate bonds in the ETF have a relatively short duration comparing to our liability, adding some Treasury bonds and Treasury futures helps to minimize the interest risk exposure by better matching the overall duration of asset and liability. The corresponding funding ratio is quite stable around 100% as indicated in Figure 15.

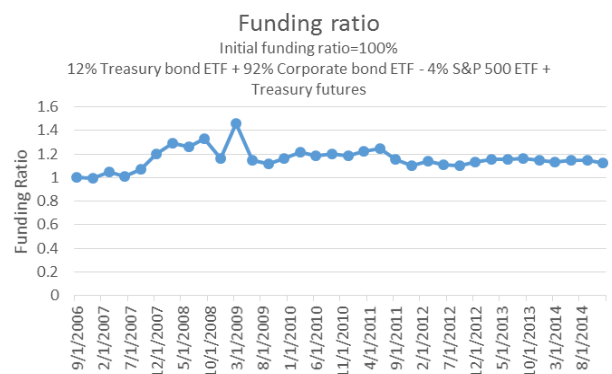

Figure 15. Funding Ratio (Initial funding ratio=100%)

4.4.2 Under-Funded Scenario

Revised Sharpe Ratio When we are suffering from a funding gap, maintaining a stable funding ratio is not enough anymore. Some amount of excess return is desired in order to solve the under-funding problem. Under this circumstance, the idea to maximize Sharpe ratio can be helpful which seeks high excess return while maintaining a low volatility. In our case, we introduce a revised Sharpe ratio as follows:

$$SR = \frac{\overline{gr}}{\sigma_{gr}} \tag{10}$$

where $\overline{gr}$ is the average growth rate of funding ratio and σ_{gr} is the corresponding volatility. Both of them are

Funding ratio	Treasury ETF	Corporate ETF	S&P 500 ETF	Revised Sharpe ratio	Mean return (quarterly)
100%	-0.24	-1	2.24	0.2848	0.0448
95%	-0.16	-1	2.16	0.2903	0.0439
90%	-0.04	-1	2.04	0.2965	0.0322
85%	-0.04	-0.84	1.88	0.3035	0.0402
80%	-0.08	-0.6	1.68	0.3121	0.0378
75%	-0.12	-0.36	1.48	0.3225	0.0355

Table 1. Investment Strategies and Initial Funding Ratios

quarterly data. Since the liability is dynamic, we cannot simply use the return and volatility of asset to calculate Sharpe ratio. Instead, the funding ratio indicates the matching level of asset and liability, and a growing funding ratio with low volatility is desired for the pension plan.

Construction of Optimal Portfolio The basic idea to calculate the optimal portfolio is similar to the fully-funded scenario, while the objective is to maximize the revised Sharpe ratio. The investment strategies based on the initial funding ratio is listed in table 1 .

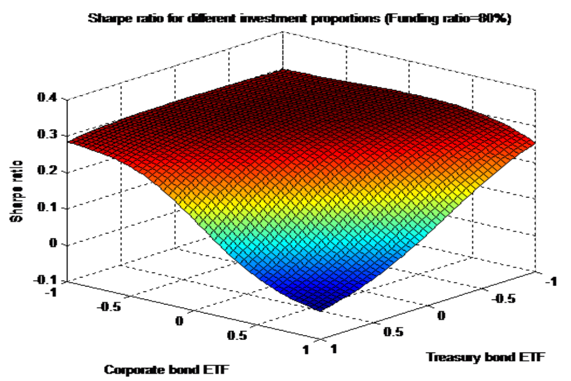

Figure 16. Sharpe ratio for different investment portfolio (Initial funding ratio=80%)

The major strategy is to short bonds and long stocks, as the equity market outperformed fixed income market during the past years. Such a leveraged strategy is very helpful for profit generation. We take the 80% funding ratio scenario as an example. From the revised Sharpe ratio surface in Figure 16, short position in both Treasury and corporate ETF will lead to a high revised Sharpe ratio.

As the funding ratio goes higher, we can spare more money to achieve the duration matching objective by investing in Treasury futures, which makes it possible to short more bond ETF and create a higher leverage to long equity. Although the Sharpe ratio goes a little bit lower than the under-funded scenario, the mean return increases significantly.

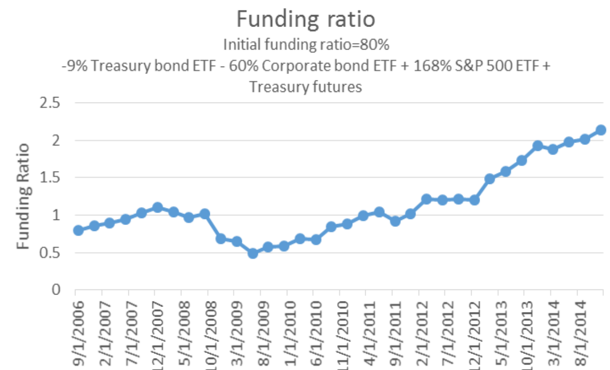

Figure 17. Funding ratio (Initial funding ratio=80%)

Therefore, in the under-funded scenario, we strongly suggest a long position in equity market. More volatility and a higher potential profit will help us to fix the funding gap more quickly according to Figure 17. The short position in bonds is an applicable method to create some leverage, which can help to speed up the healing process.

5. Conclusion

Based on different initial funding ratios, the investment strategy may vary for pension funds to successfully fulfill the obligation to beneficiaries.

If we start from a fully-funded scenario, the primary concern is to maintain a exact and stable match of the asset and liability. It suffices if we can successfully hedge most of the risks for liability and stay fully-funded. Most of our investments will focus on fixed-income products to pursue stability and duration-matching is the major method.

If we suffer from a under-funded problem at the beginning, however, more excess profit will be desired to eliminate the funding gap. Some investment in equity market will be advised to generate a higher potential return. Meanwhile, the volatility always deserves a special attention as the astonishing growth of stocks is unlikely to last for long term. The revised Sharpe ratio deals with this issue properly. Based on this index, we can select the optimal investment strategy with high return while maintaining a relatively low volatility.

References

Bruder, B., Guillaume, J., and Guillaume, L. Beyond Liability-driven Investment: New Perspectives on Defined Benefit Pension Fund Management, 2010.

Dickson, David CM, et al. Actuarial mathematics for life contingent risks. Cambridge University Press, 2013.

Finnerty, John D., and Dwight Grant. Testing hedge effectiveness under SFAS 133: CPA Journal 73.4, 2003.

Winkelmann, K., McDermott S., et al. "Liability-Driven Investment Policy: Structuring the Hedging Portfolio." Goldman Sachs Asset Management Working Paper, 2007

Meder, Aaron, and Renato Staub. "Linking Pension Liabilities to Assets." UBS Global Asset Management Working Paper, March 17 (2006).

Black, Fischer. Should you use stocks to hedge your pension liability? Financial Analysts Journal (1989): 10-12.

iShares by BlackRock ETF data base.

Appendix

Mathematical Derivation in Liability Modeling

Let (x) denote a life with the age of x, where $x \geq 0$.
We first define a random variable T_x to be the future lifetime of (x), which is equivalent to say, $x+T_x$ represents the age-at-death random variable for (x). Let F_x be the distribution function of T_x, so that

$$F_x(t) = Pr[T_x \leq t] \tag{11}$$

We make following assumption for all $x \geq 0$ and for all $t > 0$,

$$Pr[T_x \leq t] = Pr[T_0 \leq x+t | T_0 > x] = \frac{Pr[x < T_0 \leq x+t]}{Pr[T_0 > x]}$$

That is,

$$F_x(t) = \frac{F_0(x+t) - F_0(x)}{1 - F_0(x)} \tag{12}$$

We then define the survival function S_x as

$$S_x(t) = 1 - F_x(t) = Pr[T_x > t] \tag{13}$$

Thus, $S_x(t)$ represents the probability that (x) survives for at least t years.
Equation (13) becomes,

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} \tag{14}$$

In actuarial mathematics, the force of mortality represents the instantaneous rate of mortality at a certain age measured on an annualized basis. We denote the force of mortality at age x by μ_x and define it as

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} Pr[T_0 \leq x+dx | T_0 > x] \tag{15}$$

From equation (12) we see that an equivalent way of defining μ_x is

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} Pr[T_x \leq dx] \tag{16}$$

which can be written in terms of the survival function S_x as

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx}(1 - S_x(dx)) \tag{17}$$

Therefore, equation (16) gives the approximation,

$$\mu_x dx \approx Pr[T_0 \leq x+dx | T_0 > x] \tag{18}$$

For very small dx, we can interpret $\mu_x dx$ as the probability that a life who has attained age x dies before attaining age $x+dx$, thus relating the force of mortality to the survival function,

$$S_x(dx) = \frac{S_0(x+dx)}{S_0(x)} \tag{19}$$

Plug into equation (18) we get,

$$\begin{aligned}\mu_x &= \lim_{dx \to 0^+} \frac{1}{dx}\left(1 - \frac{S_0(x+dx)}{S_0(x)}\right) \\ &= \frac{1}{S_0(x)} \lim_{dx \to 0^+} \frac{S_0(x) - S_0(x+dx)}{dx} \\ &= \frac{1}{S_0(x)}\left(-\frac{d}{dx}S_0(x)\right)\end{aligned}$$

Thus,

$$\mu_x = \frac{-1}{S_0(x)}\frac{d}{dx}S_0(x) = -\frac{d}{dx}logS_0(x) \tag{20}$$

and integrating this identity over $(0, y)$ yields

$$\int_0^y \mu_x dx = -(logS_0(y) - logS_0(0)) \tag{21}$$

As $logS_0(0) = logPr[T_0 > 0] = log1 = 0$, we obtain

$$S_0(y) = exp\left\{-\int_0^y \mu_x dx\right\} \tag{22}$$

from which it follows that

$$\begin{aligned}S_x(t) = \frac{S_0(x+t)}{S_0(x)} &= exp\left\{-\int_x^{x+t} \mu_r dr\right\} \\ &= exp\left\{-\int_0^t \mu_{x+s} ds\right\}\end{aligned} \tag{23}$$

With Makeham's Law, we have

$$\mu_x = A + Bc^x \tag{24}$$

From equation (24) we get,

$$\begin{aligned}S_x(t) &= exp\left\{-\int_0^t \mu_{x+s} ds\right\} \\ &= exp\left\{-\int_0^t (A + Bc^{x+s}) ds\right\} \\ &= exp\left\{-\left(As + \frac{B}{logc}c^{x+s}\right)\Big|_{s=0}^{s=t}\right\} \\ &= exp\left\{-At - \frac{B}{logc}(c^{x+t} - c^x)\right\} \\ &= e^{-At} exp\left\{-\frac{B}{logc}c^x(c^t - 1)\right\}\end{aligned} \tag{25}$$

Thus, it can be written as,

$$S_x(t) = s^t g^{c^x(c^t-1)} \tag{26}$$

where $s = e^{-A}$ and $g = exp\{-\frac{B}{logc}\}$, and here we take the standard values $A = 0.00022$, $B = 2.7 \times 10^{-6}$ and $c = 1.124$.