Variable Volatility & Financial Failure

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Abstract

Structural models of corporate default (eg. Merton's model) typically impose a rigid parametric specification on the volatility of the firm's assets. This approach fails to recognize that management can exert some limited control on the level of the assets' risk at every possible level of their firm's default probability. In this paper, we assume that management chooses the assets' volatility level as a non-parametric function of the firms' risk-neutral default probability (RNDP). We develop closed form formulas which relate RNDP and equity value to this asset volatility function and to asset price. We also show how to explicitly determine the implied RNDP and the implied asset value from the market price of the equity and from the market prices of co-terminal calls written on the equity. Remarkably, the RNDP formula is independent of both the initial asset level and the debt level.

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1 Introduction

Invert, always invert – Carl Gustav Jacob Jacobi

On October 14 1997, the Royal Swedish Academy of Sciences announced the award of the Nobel prize in Economic Sciences to Professors Merton and Scholes for a new method to determine the value of derivatives. In the press release, the academy observed that:

> The value of the stock, preferred shares, loans, and other debt instruments in a firm depends on the overall value of the firm in essentially the same way as the value of a stock option depends on the price of the underlying stock. The laureates had already observed this in their articles published in 1973, thereby laying the foundation for a unified theory of the valuation of corporate liabilities.

The idea that the equity of a firm can be treated as a call option written on the assets of the firm appeared in the seminal paper by Black and Scholes [3]. The idea first appeared in Merton[10] and hence is commonly referred to as the Merton model. Well before 1997, the Merton[11] model had become a market standard for valuing corporate securities. The Merton model and its variations are referred to as structural models, as they attempt to endogenize corporate default.

The standard approach to structural models starts by specifying the dollar volatility of the firm's assets as a parametric function of the asset's value. For example, in Merton's model, the dollar volatility of the firm's assets is modeled as positively proportional to the asset's value. Any such parametric specification fails to recognize the extent of the latitude that managers have in setting the asset mix and hence the asset volatility in every possible scenario. In principle, the instantaneous dollar volatility can depend on the path taken by the asset's value as well as on a wide host of additional variables.

As an approximation to such a complicated reality, our approach only considers firms for which the assets' volatility depends only on the firm's risk-neutral default probability (RNDP). The dependence on RNDP rather than real-world default probability is based on the easier observability of the former and on the belief that market risk aversion is also relevant in setting the asset's volatility level. The restriction to the class of asset volatilities which depend only on the firm's RNDP is actually derived as a consequence of a more fundamental property.

To describe this property, define the net asset value (NAV) as the difference between the market value of the firm's assets and the *book* value of the firm's liabilities. This definition of NAV allows NAV to become negative, which cannot happen if NAV were instead defined using the market value of the firm's liabilities. In this paper, we assume that a firm has a single issue of zero coupon debt. As a result, default can occur only at the debt's maturity date and default occurs at this time if and only if the firm's NAV is negative then.

Consider two firms who start out solvent and who must each retire their debt in one quarter. Suppose that the two firms are identical, except that the second firm's NAV is twice that of the first. Clearly, the second firm has a lower RNDP than the first. Now suppose we start pushing out the maturity date of the second firm's debt. Since the variance of the second firm's terminal asset value increases with the horizon, the second firm's RNDP will start rising. When the second firm's time to maturity is a sufficiently large multiple of one quarter, the second firm will have the same RNDP as the first firm. In this paper, the multiple will be four, i.e. when the second firm's debt matures in one year, then this firm has the same default probability as the first firm.

More generally, in this paper, multiplying NAV by a factor $m > 0$ leaves default probability unchanged if the horizon is also multiplied by m^2. Put another way, RNDP depends only on the *normalized* NAV, where we normalize NAV by dividing it by the square root of the time to maturity. We refer to this normalized NAV as distance to default. We start by assuming that the firm's asset volatility depends only on distance to default. We then show as a consequence that the firm's RNDP is given by an explicit function, which depends only on distance to default. This function is invertible and hence the firm's asset volatility depends only on RNDP in our class of models.

Our approach produces a risk-neutral default probability (DP) rather than a real-world DP. When the underlying assets have a positive risk premium, our RNDP will exceed the real world DP. However, when the forecast horizon is short, say one year or less, then the upward bias of our RNDP will be small since the influence of drift on this calculation over short horizons is negligible.

When asset volatility depends only on RNDP, we also show that one can derive a closed form formula relating the equity value to the asset value. Our explicit valuation formulas for failure probability and for equity value hold when the function relating asset volatility to distance to default is free to be any function in a large class, rather than constrained by a particular functional form with a small number of free parameters.

While the freedom to choose the asset volatility function is an undeniably important advantage to the modeler in achieving realism, it is unclear ex ante how a modeler should choose this function. Fortunately, when calls trade on the equity of the firm, one can observe a related volatility function. In particular, market prices of co-terminal equity calls and calendar spreads can be used to determine the function relating the stock's volatility to the stock price. Practitioners routinely refer to this function as the local volatility smile. In this paper, we present explicit formulas relating the implied RNDP and the implied asset value to this local volatility smile and to the market price of the underlying stock. To calculate the implied RNDP and the implied asset value in our framework, one also needs to know the time to the debt maturity. However, to calculate the implied RNDP, one does not need to know the level of the debt, nor does one need to know the value of the firm's assets. In essence, stock and options' prices are sufficient statistics for these two inputs, bypassing the need to observe them.

Hence, the contribution of this paper is twofold. First, given a function relating asset volatility to distance to default, we present an explicit formula relating risk-neutral default probability to (just) distance to default. In this context, we also present an explicit formula relating stock value to the excess of the assets over the promised debt payment and to the time to maturity of the debt. Both formulas require as inputs the market value of the firm's assets and the face value of the firm's debt. Our second contribution recognizes that these inputs are not directly observable. We take as given a complete set of equity option prices across all strikes at one maturity. We assume that one can also observe infinitessimal calendar spreads of these calls at this maturity. We then show that one can determine a local volatility function relating the stock's dollar volatility to the stock price. This local volatility function is used in an explicit formula relating the risk-neutral default probability to just the stock price and the time to maturity of the debt. This formula is independent of both the level of the debt and the value of the firm's assets. In this context, we also present an explicit formula relating the value of the firm's assets to the stock price and the time to maturity of the debt. In contrast to the RNDP formula, this formula uses the initial asset value and the debt level as inputs. While it is now standard practice to calibrate valuation models to a volatility smile, our approach appears to be the first instance where the calibration is explicit.

An overview of this paper is as follows. In the next section, we review two well known structural models in order to extract common elements. In the following section, we propose a new structural model where the assets' volatility is a given non-parametric function of just distance to default. We show that this assumption leads to an explicit formula for the RNDP and also to the conclusion that asset volatility just depends on default probability. We also present an explicit equity valuation formula in this context. In the next section, we show how a complete set of option prices across all strikes at one maturity can be used along with calendar spreads to determine a local volatility smile linking the instantaneous stock volatility to the stock price. In the penultimate section, we present explicit formulas for the implied RNDP and for the implied asset value. Both formulas depend on the local volatility function, the stock price, and the time to debt maturity. The RNDP formula depends only on these variables, while the asset value formula further depends on the difference between the initial asset value and the debt level. The final section summarizes the paper and makes suggestions for future research.

2 Review of Some Structural Models

In this section, we review two well known structural models in an effort to find common elements.

2.1 Merton Model

The first structural model was proposed by Merton [10]. The Merton model assumes that the value of the firm's assets follows geometric Brownian motion under the real world probability measure $\mathbb{P}$. In particular, in the Merton model, the $\mathbb{P}$ dynamics of the asset value A are given by solving the following stochastic differential equation:

$$dA_t = \mu A_t dt + \eta(A_t) dB_t, \qquad t \geq 0, \tag{1}$$

where the proportional drift rate μ is constant and where B is a $\mathbb{P}-$standard Brownian motion. In (1), the coefficient of the Brownian increment dB_t is $\eta(A_t)$, which practitioners call either the normal volatility function or the dollar volatility function. In the Merton model, the normal volatility of the firm's assets is assumed to be positively proportional to the asset's value:

$$\eta(A) = \sigma A, \qquad A > 0,$$

where σ is a positive constant.

Assuming zero interest rates, the absence of arbitrage implies the existence of a risk-neutral measure $\mathbb{Q}$ such that the prices of all non-dividend paying assets are martingales. Let M be the maturity date of the zero coupon bond that the firm has issued. Since the Merton model assumes that the firm's assets do not emit any cash flows to debt or equity over $[0, M)$, the risk-neutral dynamics of the asset value A over this time interval are simply given by:

$$dA_t = \sigma A_t dW_t, \qquad t \in [0, M], \tag{2}$$

where W is a $\mathbb{Q}-$standard Brownian motion.

Let $L > 0$ be the fixed face value of the firm's zero coupon debt maturing at M. Let $F^m(A, t; L, M)$ be the function relating the failure probability F to the state variables $A > 0$ and $t \geq 0$ and to the parameters $L > 0$ and $M \geq t$. The superscript m reminds us of the Merton model, while the last two arguments of F^m are the fixed constants L and M, representing the debt level and the debt maturity date respectively. In the Merton model, the risk-neutral failure probability F^m is given by the following explicit formula:

$$F^m(A, t; L, M) = N(-d_2(A, t; L, M)), \qquad A > 0, t \in [0, M), \tag{3}$$

where:

$$N(d) \equiv \int_{-\infty}^{d} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$$

denotes the standard normal distribution function, and where:

$$d_2(A, t; L, M) = \frac{[\ln A - \sigma^2(M - t)/2] - \ln L}{\sigma\sqrt{M - t}}. \tag{4}$$

In Merton's model:

$$\ln(A) - \sigma^2(M - t)/2 = E^{\mathbb{Q}}[\ln A_M | A_t = A]. \tag{5}$$

Hence, the failure probability can be expressed as:

$$F^m(A, t; L, M) = f(z^m(A, t; L, M); \sigma), \qquad A > 0, t \in [0, M), \tag{6}$$

where:

$$f(z^m; \sigma) \equiv N\left(-\frac{z^m}{\sigma}\right), \tag{7}$$

and:

$$z^m(A, t; L, M) = \frac{E^{\mathbb{Q}}\left[\ln A_M | A_t = A\right] - \ln L}{\sqrt{M - t}} = \frac{E^{\mathbb{Q}}\left[\ln A_M | A_t = A\right] - \ln L}{\text{std}^{\mathbb{Q}}[W_M | W_t = 0]}. \tag{8}$$

When the dependence of $E^{\mathbb{Q}}\left[\ln A_M | A_t = A\right]$ on A can be directly observed, (6) indicates that the failure probability arises by evaluating the parametric function $f(z^m; \sigma)$ at the point $z^m(A, t; L, M)$ which is independent of σ. In the next subsection, we find that the failure probability in a different model has the same structure.

We refer to $z^m \in \mathbb{R}$ as the geometric distance to default. It differs from d_2 in that the denominator of z^m is independent of σ. In the next section, we will introduce an even simpler measure of distance to default. Substituting (4) in (3) makes it clear that the failure probability is decreasing in the asset value:

$$F^m(A,t;L,M) = N\left(\frac{\ln L - \ln A + \sigma^2(M-t)/2}{\sigma\sqrt{M-t}}\right), \qquad A > 0, t \in [0,M). \tag{9}$$

Solving for A, the asset value can be regarded as a decreasing function of the failure probability:

$$A(F,t) = Le^{\sigma^2(M-t)/2-\sigma\sqrt{M-t}N^{-1}(F)}, \qquad F \in [0,1], t \in [0,M). \tag{10}$$

Since the normal volatility of the assets is positively proportional to the asset value, it follows that the normal volatility is a decreasing function of the failure probability:

$$\tilde{\eta}^m(F,t) \equiv \sigma A(F,t) = \sigma Le^{\sigma^2(M-t)/2-\sigma\sqrt{M-t}N^{-1}(F)}, \qquad F \in [0,1], t \in [0,M). \tag{11}$$

In the Merton model, one can explicitly relate the value of the stock to the value of the assets of the firm. Let $S^m(A,t;L,M)$ be the function relating the stock value S to the asset value A and to calendar time $t \in [0,M]$. In the Merton model, the terminal payoff to equity is that of a call option written on the assets of the firm:

$$S^m(A,M;L,M) = (A-L)^+, \qquad A > 0. \tag{12}$$

Using the Black Scholes[3] call formula, the value of the stock at any prior time is given by:

$$S^m(A,t;L,M) = AN(d_2(A,t;L,M)+\sigma\sqrt{M-t}) - LN(d_2(A,t;L,M)), \qquad A > 0, t \in [0,M). \tag{13}$$

The debt value is given by subtracting the stock value from the asset value.

In the Merton model, one can explicitly relate the value of a call on the stock to the value of the assets of the firm. In particular, Geske[7] provides a closed form solution for the compound call value. In contrast, there is no explicit formula relating the value of a stock call to the stock price. Rather, the dependence is implicit due to the difficulty in explicitly inverting (13) for A.

Hull, Nelken, and White[8] consider the problem of calibrating the parameter σ and initial state variable A_0 in Merton's model to the stock price and to the price of an equity option. One can numerically solve for these two inputs and then use Geske's formula to value an equity call. This gives equity call prices as an explicit function of asset values.

2.2 Bachelier Model

Since the publication of the Merton paper in 1974, various authors have explored alternative stochastic processes for the asset value. One of the simpler alternatives would replace the geometric Brownian motion described by (2) with an arithmetic process:

$$dA_t = \eta dW_t, \qquad t \geq 0, \tag{14}$$

where the normal volatility η is now constant. In Merton's model, the asset values are always positive, whereas in Bachelier's model, the assets are real-valued.

To justify real-valued assets, suppose we start in Merton's world where the firm's assets consist of one stock with random price $S_{1t} > 0$ and the liability is described as a promise to pay the constant $L > 0$ at the future time $M > 0$ with present value $D_t(L) > 0$. Now suppose that at $t = 0$, the firm shorts one share of a second unrelated stock with random price $S_{2t} > 0$. At some later time $t \geq 0$, one can say that this firm has assets worth the positive random amount $S_{20} + S_{1t}$ and a liability worth the positive random amount $D_t(L) + S_{2t}$. Alternatively, one can say that this firm has assets worth the real random amount $S_{20} + S_{1t} - S_{2t}$ and a liability worth the positive random amount $D_t(L)$.

Given the usefulness of real numbers, we will adopt the latter view. One does not need short stock positions to motivate real-valued assets. As pointed out by Eberlein and Madan[6], firms sometimes take risks whose worst realization can take the asset value negative. For example, a tobacco firm runs the risk that damage awards exceed assets on hand or a re-insurer runs the risk that future claims exceed reserves. For the remainder of this paper, assets will be real-valued, but we will still respect the limited liability of equity.

In Bachelier's model, the failure probability is:

$$F^b(A, t; L, M; \eta) = N\left(-\frac{z(A, t; L, M)}{\eta}\right), \qquad A \in \mathbb{R}, t \in [0, M), \tag{15}$$

where:

$$z(A, t; L, M) \equiv \frac{A - L}{\sqrt{M - t}} = \frac{E^{\mathbb{Q}}[A_M | A_t = A] - L}{\text{std}^{\mathbb{Q}}[W_M | W_t = 0]}, \qquad A \in \mathbb{R}, t \in [0, M). \tag{16}$$

We refer to $z \in \mathbb{R}$ as the normalized arithmetic distance to default. Our z differs from z^m in that z depends on the *arithmetic* distance between A and L, rather than the geometric distance. From (15), the failure probability F^b depends on the state variables A and t only through $z = \frac{A-L}{\sqrt{M-t}}$.

One can also value the stock in the Bachelier model. The payoff to the stock at maturity is still given by (12), with the domain for asset value extended to $\mathbb{R}$. In contrast, the stock value prior to maturity is given by the following call formula in Bachelier[1]:

$$S^b(A, t; L, M; \eta) \equiv (A - L)N\left(\frac{z(A, t; L, M)}{\eta}\right) + \eta\sqrt{M - t}N'\left(\frac{z(A, t; L, M)}{\eta}\right), A \in \mathbb{R}, t \in [0, M). \tag{17}$$

In Bachelier's model, asset values can be negative, but the equity value is always non-negative as usual. Suppose that the debt value is again given by subtracting the equity value from the asset value. When asset value is negative, this will result in a negative debt value. If we extend limited liability to bondholders, then a corporate bond can be valued by calculating the value of the government guarantee that neither debt nor equity be negative when asset value can be negative. For an example of such a calculation, see Eberlein and Madan[6].

It would be straightforward to determine a Geske-type formula for the value of an equity call. This compound call formula would again use a bivariate normal distribution function. The calibration methodology would be similar to that encountered with the Merton model.

3 A Non-parametric Structural Model

In the last section, we reviewed Merton's structural model and Bachelier's structural model. In both models, the failure probability just depends on the appropriate measure of distance to default. The failure probability and the equity value are both given by explicit formulas. However, both of these models describe asset volatility with a single parameter. The main thesis of this paper is that models which impose a rigid specification of asset volatility also exert a pernicious influence on the calculation of default probability. Consider two firms which are identical in every way except for management control over the asset mix. Suppose that management of the first firm is easily able to adjust asset mix and hence asset volatility in response to changing fortunes, while the second firm has a fixed asset mix and hence a fixed asset volatility. Suppose that the average volatility over all paths for the two firms is the same. In models like Merton and Bachelier, the default probabilities will also be the same since such models fail to appreciate the greater influence of downside volatility on default probability than upside volatility. This issue is compounded in the Merton model by the well known leverage effect which is the empirical observation that lognormal volatilities are declining in asset price. The Bachelier model roughly captures this leverage effect, but is still too inflexible to fully encompass management's ability to exert control on the asset volatility as the probability of default changes.

In this section, we propose an alternative structural model which generalizes Bachelier's model. In our model, the dollar volatility of the assets is described by a free function rather than a free parameter. The independent variable governing asset volatility is taken to be the risk-neutral default probability (RNDP). All of the other assumptions in the previous section remain in force.

While one could attempt to treat the state variable as an RNDP living inside $[0, 1]$, it is easier from a mathematical standpoint to treat asset volatility and RNDP as functions of the normalized arithmetic distance to default variable $z(A, t; L, M) \equiv \frac{A-L}{\sqrt{M-t}}$, which is real-valued. Hence, we start by assuming that asset volatility η is an arbitrary function $\eta(z)$ of just distance of default. We then derive a closed form formula that relates RNDP to just distance to default. By inverting this formula, we can show that asset volatility just depends on RNDP in the type of models we are considering. For this class of models, we also develop a closed form formula which relates the equity value to the asset value. In the next section, we also show how to calibrate the asset volatility function $\eta(z)$ to market prices of call options written on the equity. As a result, these call prices can be used to infer the market's belief about where management will set the asset volatility at each possible level of default probability. We also give an explicit formula relating the RNDP to the stock price and to the given array of call prices.

It is well known that the Moody's KMV model also relates default probability to distance to default. Our approach is similar to theirs, but has three important differences. First, the dependent variable in the Moody's KMV model is the real-world default probability, rather than the risk-neutral one. Second, their measure of distance to default differs from ours in that it is based on a geometric view of asset shocks rather than an arithmetic view. Finally, the Moody's KMV model directly asserts a relationship between default incidence and distance to default, whereas we derive this relationship as a consequence of a formal dynamical model which asserts a functional relationship between asset volatility and distance to default. To determine the link between real-world default probability and geometric distance to default, the Moody's KMV model makes a stationarity assumption thereby allowing them to use historical data to calibrate the functional relationship. In the next section, we will show how option prices can be used in our approach to obtain the link between asset volatility and distance to default. We thereby also obtain the link between RNDP and arithmetic distance to default, without having to rely on the stationarity assumption. The cumulative effect of the three differences has implications for usage. The Moody's KMV approach is relevant for measuring default risk, whereas our approach is more appropriate for valuing corporate claims e.g. stock, bonds, CDS, and options.

In our non-parametric structural model, our new risk-neutral process[1] for asset value is given by:

$$dA_t = \eta(Z_t)dW_t, \qquad t \in [0, M), \tag{18}$$

where the argument of the normal volatility function $\eta(\cdot)$ is a real-valued stochastic process Z_t defined by:

$$Z_t \equiv \frac{A_t - L}{\sqrt{M-t}}, \qquad t \in [0, M). \tag{19}$$

We refer to Z as simply "distance to default" with its normalized and arithmetic nature being understood. We assume that the volatility function $\eta(z)$ is bounded away from zero. As the debt nears its maturity, distance to default diverges almost surely to either positive infinity or negative infinity. However, the dollar volatility of the assets will not explode if the volatility function $\eta(z)$ asymptotes to a finite nonzero constant in each case. We now assume that the volatility function has this regularizing behavior.

We regard the distance to default Z_t as a proxy for the risk-neutral default probability F_t. From now on, we refer to F as the firm's failure probability with its risk-neutral nature being understood. When the function $\eta(z)$ is known, we later show that z is in fact a known function $\tilde{z}(f)$ of the failure probability f. As a result, the firm's asset volatility becomes known as a composite function of the firm's failure probability:

$$dA_t = \tilde{\eta}(F_t)dW_t, \qquad t \in [0, M), \tag{20}$$

where $\tilde{\eta}(f) \equiv \eta(\tilde{z}(f))$.

Our assumed dependence of asset volatility on distance to default can be motivated financially. One can think of Z as measuring the number of standard deviations that the asset value A is above the debt level L in the Bachelier model when the dollar volatility is constant at $\eta = 1$. Since default occurs at the debt maturity date M when $Z_M < 0$, it is reasonable to assume that managers pay attention to the Z process in managing the firm's assets. When z is positive, it seems reasonable that the function η would be increasing in z, so that reductions in distance to default would be accompanied by reductions in the dollar volatility of the firm's assets. On the other hand, when z is negative, the asset substitution problem suggests that managers might prefer to have larger volatility, i.e. gamble to increase the probability of avoiding a default. Since our analysis allows η to be non-monotonic, we can proceed without knowing the sign of the slope. We will later show how equity options can be used to infer the η function.

[1] Our SDE (18) is similar to one that can be found in Madan and Yor[9].

For the remainder of this paper, we explore the consequences of the assumption that the asset volatility just depends on the distance to default variable z. In both the Merton model and the Bachelier model, recall that the failure probability is just a function of the appropriate notion of distance to default. We choose an arithmetic distance measure on the grounds that it is simpler than a geometric distance measure and to take mathematical advantage of the fact that the equity's payoff is $(A_M - L)^+$, rather than $(\ln A_M - \ln L)^+$.

3.1 Calculating Risk-Neutral Default Probability

We first turn our attention to the calculation of the risk-neutral default probability. Let $F(A, t; L, M)$ be the failure probability, considered as a function of the asset value A and calendar time t. The failure probability solves the following partial differential equation (PDE):

$$\frac{1}{2}\eta^2\left(\frac{A-L}{\sqrt{M-t}}\right)\frac{\partial^2}{\partial A^2}F(A,t;L,M) + \frac{\partial}{\partial t}F(A,t;L,M) = 0, \qquad A \in \mathbb{R}, t \in [0, M), \tag{21}$$

subject to the following terminal condition:

$$F(A, M; L, M) = 1(A < L), \qquad A \in \mathbb{R}. \tag{22}$$

Motivated by the Bachelier model, we consider changing the independent variable in the PDE. Let:

$$f(z) \equiv F(A, t; L, M), \tag{23}$$

where the new independent variable is:

$$z \equiv \frac{A-L}{\sqrt{M-t}}. \tag{24}$$

Differentiating (23) w.r.t. A:

$$\frac{\partial}{\partial A}F(A,t;L,M) = \frac{f'(z)}{\sqrt{M-t}}, \tag{25}$$

since $\frac{\partial}{\partial A}z = \frac{1}{\sqrt{M-t}}$ from (24). Differentiating (25) w.r.t. A:

$$\frac{\partial^2}{\partial A^2}F(A,t;L,M) = \frac{f''(z)}{M-t}. \tag{26}$$

Differentiating (23) w.r.t. t:

$$\frac{\partial}{\partial t}F(A,t;L,M) = \frac{A-L}{2(\sqrt{M-t})^3}f'(z) = \frac{z}{2(M-t)}f'(z), \tag{27}$$

from (24).

Substituting (26) and (27) in the PDE (21) governing F reduces it to the following second order linear ordinary differential equation (ODE):

$$\eta^2(z)f_{zz}(z) + zf_z(z) = 0, \qquad z \in \mathbb{R}. \tag{28}$$

Recall that when $t \uparrow M$, the distance to default variable z explodes to $\pm\infty$ almost surely. In our new coordinate system, the terminal condition (22) maps to the following pair of Dirichlet boundary conditions:

$$\lim_{z \downarrow -\infty} f(z) = 1 \text{ and } \lim_{z \uparrow \infty} f(z) = 0. \tag{29}$$

The unique solution to the two point boundary value problem (28) and (29) can be represented as:

$$f(z) = 1 - \frac{1}{b_f} \int_{-\infty}^{z} e^{-\int_0^y \frac{x}{\eta^2(x)} dx} dy, \qquad z \in \mathbb{R}, \tag{30}$$

where the positive normalizing constant b_f is given by:

$$b_f = \int_{-\infty}^{\infty} e^{-\int_0^y \frac{x}{\eta^2(x)} dx} dy. \tag{31}$$

As a result, the firm's failure probability $F(A, t; L, M)$ is a function $f(z)$ given by (30) and (31). One can check that our solution for F solves both the PDE (21) and the terminal condition (22). Since the solution to this terminal value problem is unique, we have found it. Since the failure probability in (30) depends only on z, we refer to (30) as the isotherm formula for RNDP. For particular choices of the volatility function $\eta(\cdot)$ the integrals in this isotherm formula can be expressed in terms of special functions. Setting $\eta(\cdot) = \eta$ in (30) leads to the Bachelier formula (15) for the failure probability, and hence (30) is a generalization of this formula. If $\eta(z)$ were piecewise constant, it would be straightforward to calculate $f(z)$ in closed form using the standard normal CDF $N(\cdot)$. When the integrals in (30) can't be expressed analytically, one can either use numerical quadrature or else use finite differences on the two point boundary value problem (28) and (29).

Notice that the integrand and the normalizing constant b_f in (30) are both positive. As a result, the failure probability $F(A, t; L, M) = f(z)$ is a decreasing function of just the distance to default $z = \frac{A-L}{\sqrt{M-t}}$. Let $\tilde{z}(f)$ be the inverse function. Then, distance to default $\tilde{z}(f)$ is decreasing in the failure probability f and depends only on this variable. It follows that the assets' dollar volatility just depends on the failure probability in our class of models:

$$dA_t = \tilde{\eta}(F_t) dW_t, \qquad t \in [0, M), \tag{32}$$

where $F_t = f\left(\frac{A_t - L}{\sqrt{M-t}}\right)$ is the failure probability at time t and $\tilde{\eta}(f) \equiv \eta(\tilde{z}(f))$ is the assets' dollar volatility, written as a function of the firm's failure probability. In principle, the sole dependence of asset volatility on failure probability is a testable implication for our family of models. Our whole model family can be rejected if one repeatedly found that asset volatility differed on two distinct dates that happened to lead to the same RNDP.

3.2 Valuing Equity

We now turn our attention to the problem of valuing equity when the asset volatility just depends on the failure probability. Let $S(A, t; L, M)$ be the stock value function:

$$S(A, t; L, M) = E^{\mathbb{Q}}[(A_M - L)^+ | A_t = A], \qquad A \in \mathbb{R}, t \in [0, M). \tag{33}$$

This function solves the same PDE (21) as the failure probability:

$$\frac{1}{2}\eta^2\left(\frac{A-L}{\sqrt{M-t}}\right)\frac{\partial^2}{\partial A^2}S(A,t;L,M)+\frac{\partial}{\partial t}S(A,t;L,M)=0, \qquad A\in\mathbb{R}, t\in[0,M). \tag{34}$$

However, the stock value is subject to a different terminal condition:

$$S(A,M;L,M)=(A-L)^+, \qquad A\in\mathbb{R}. \tag{35}$$

Consider the following change of both dependent and independent variables:

$$s(z)\equiv\frac{S(A,t;L,M)}{\sqrt{M-t}}, \qquad A\in\mathbb{R}, t\in[0,M), \tag{36}$$

where as before:

$$z\equiv\frac{A-L}{\sqrt{M-t}} \qquad A\in\mathbb{R}, t\in[0,M). \tag{37}$$

Since the stock is priced at zero when the firm is in default, the new dependent variable can be described as the distance to default of the stock. To avoid confusion with the asset's distance to default z, we will refer to s as the normalized stock price. Re-arranging (36) yields:

$$S(A,t;L,M)=\sqrt{M-t}s(z) \qquad A\in\mathbb{R}, t\in[0,M). \tag{38}$$

Differentiating (38) w.r.t. A:

$$\frac{\partial}{\partial A}S(A,t;L,M)=s'(z) \qquad A\in\mathbb{R}, t\in[0,M), \tag{39}$$

since $\frac{\partial}{\partial A}z=\frac{1}{\sqrt{M-t}}$ from (37). Differentiating (39) w.r.t. A:

$$\frac{\partial^2}{\partial A^2}S(A,t;L,M)=\frac{s''(z)}{\sqrt{M-t}} \qquad A\in\mathbb{R}, t\in[0,M). \tag{40}$$

Differentiating (38) w.r.t. t:

$$\frac{\partial}{\partial t}S(A,t;L,M)=-\frac{s(z)}{2\sqrt{M-t}}+s'(z)\frac{z}{2\sqrt{M-t}} \qquad A\in\mathbb{R}, t\in[0,M). \tag{41}$$

Substituting (40) and (41) in the PDE (34) yields the following ODE:

$$\eta^2(z)s''(z)+zs'(z)-s(z)=0, \qquad z\in\mathbb{R}. \tag{42}$$

Recall again that when $t\uparrow M$, the distance to default variable z explodes to $\pm\infty$ almost surely. In our new coordinate system, the terminal condition (35) maps to the following pair of Dirichlet boundary conditions:

$$\lim_{z\downarrow-\infty}s(z)=0 \text{ and } \lim_{z\uparrow\infty}s(z)\sim z, \qquad z\in\mathbb{R}. \tag{43}$$

It is well known that there exists a unique solution $s(z)$ to this two point boundary value problem. We now show that this solution can be represented explicitly via quadrature.

Equation (42) is a second order linear homogeneous ODE. As a result, the general solution is a linear combination of any two linearly independent solutions. It is easily verified that the identity map $s_1(z) = z$ is a solution to this homogeneous ODE. Using reduction of order, we obtain a second solution of the homogeneous ODE (42):

$$s_2(z) \equiv z \int_{-\infty}^{z} \frac{e^{-\int_0^y \frac{x}{\eta^2(x)}dx}}{y^2} dy, \qquad z < 0. \tag{44}$$

We restrict z to the negative reals out of concern that the integral has a singularity at $z = 0$ for some volatility functions $\eta(\cdot)$.

By computing their Wronskian, one can verify that the two solutions $s_1(z)$ and $s_2(z)$ are linearly independent. Hence, for $z < 0$, the general solution of the ODE has the form:

$$G(z) = B_1 z + B_2 z \int_{-\infty}^{z} \frac{e^{-\int_0^y \frac{x}{\eta^2(x)}dx}}{y^2} dy, \tag{45}$$

where B_1 and B_2 are constants. Integrating (45) by parts:

$$\begin{aligned} u &= e^{-\int_0^y \frac{x}{\eta^2(x)}dx} \qquad dv = \tfrac{1}{y^2}dy \\ du &= -\frac{y}{\eta^2(y)} e^{-\int_0^y \frac{x}{\eta^2(x)}dx} dy \qquad v = -\tfrac{1}{y}, \end{aligned} \tag{46}$$

leads to the general solution of the ODE for $z < 0$ having the form:

$$G(z) = B_1 z - B_2 z \left[\frac{e^{-\int_0^z \frac{x}{\eta^2(x)}dx}}{z} + \int_{-\infty}^{z} \frac{e^{-\int_0^y \frac{x}{\eta^2(x)}dx}}{\eta^2(y)} dy \right]. \tag{47}$$

Suppose we guess that (47) represents the general solution for all $z \in \mathbb{R}$. The integral is now well defined since $\eta^2(\cdot)$ can't vanish. As $z \downarrow -\infty$, the two terms in square brackets in (47) both vanish. As a result, the lower boundary condition in (43) can only be met if $B_1 = 0$. Letting $b_s \equiv -\frac{1}{B_2}$, the general solution (47) is now:

$$G(z) = \frac{e^{-\int_0^z \frac{x}{\eta^2(x)}dx}}{b_s} + zI(z), \qquad z \in \mathbb{R}, \tag{48}$$

where:

$$I(z) \equiv \int_{-\infty}^{z} \frac{e^{-\int_0^y \frac{x}{\eta^2(x)}dx}}{b_s \eta^2(y)} dy, \qquad z \in \mathbb{R}. \tag{49}$$

Since the first term in (48) vanishes as $z \uparrow \infty$, the upper boundary condition in (43) will be satisfied if $I(\infty) = 1$, which from (49) implies that:

$$b_s = \int_{-\infty}^{\infty} \frac{e^{-\int_0^y \frac{x}{\eta^2(x)} dx}}{\eta^2(y)} dy. \tag{50}$$

When this positive value of b_s is used in (48), we arrive at the following simple formula for the normalized stock price:

$$s(z) = \frac{1}{b_s} \left[e^{-\int_0^z \frac{x}{\eta^2(x)} dx} + z \int_{-\infty}^{z} \frac{e^{-\int_0^y \frac{x}{\eta^2(x)} dx}}{\eta^2(y)} dy \right], \qquad z \in \mathbb{R}. \tag{51}$$

This function is positive for all z. The first derivative is simply:

$$s'(z) = \int_{-\infty}^{z} \frac{e^{-\int_0^y \frac{x}{\eta^2(x)} dx}}{b_s \eta^2(y)} dy, \qquad z \in \mathbb{R}, \tag{52}$$

which is positive, while the second derivative is:

$$s''(z) = \frac{e^{-\int_0^z \frac{x}{\eta^2(x)} dx}}{b_s \eta^2(z)}, \qquad z \in \mathbb{R}, \tag{53}$$

which is also positive.

Using (38), we arrive at our stock valuation formula:

$$S(A, t; L, M) = \sqrt{M - t} s\left(\frac{A - L}{\sqrt{M - t}}\right) = \frac{1}{b_s} \left[\sqrt{M - t} e^{-\int_0^z \frac{x}{\eta^2(x)} dx} + (A - L) \int_{-\infty}^{z} \frac{e^{-\int_0^y \frac{x}{\eta^2(x)} dx}}{\eta^2(y)} dy \right], \tag{54}$$

from (37) and (51), where z is given in (37) and b_s is given in (50). One can check that our solution for S solves both the PDE (34) and the terminal condition (35). Since the solution to this terminal value problem is unique, we have found it. For particular choices of the volatility function $\eta(\cdot)$ the integrals in (54) can be expressed in terms of special functions. Setting $\eta(\cdot) = \eta$ in (51) leads to the Bachelier call formula (17) and hence (54) is a generalization of the Bachelier call formula. If $\eta(z)$ were piecewise constant, it would be straightforward to calculate $s(z)$ in closed form using the standard normal CDF $N(\cdot)$. When the integrals in (51) can't be expressed analytically, one can either use numerical quadrature or else use finite differences on the two point boundary value problem (42) and (43).

3.3 Interpreting Stock Gamma

Let $X_t = A_t - L$ be the excess at time t of the current asset value A_t over the promised debt payment L. Let $\tilde{S}(X,t) = \sqrt{M-t}s(z)$ be the stock price as a function of this excess and time. Using the ODE (42) for s, it is easily shown that $\tilde{S}$ solves an ODE in X for every $t \in [0, M)$:

$$\eta^2\left(\frac{X}{\sqrt{M-t}}\right)(M-t)\tilde{S}_{11}(X,t) = -[X\tilde{S}_1(X,t) - \tilde{S}(X,t)], \qquad X \in \mathbb{R}. \tag{55}$$

The excess X_t is the value at time t of a forward contract written on the assets with maturity date M and delivery price L. The RHS of (55) is the dollars lent at time t when replicating equity by dynamically trading this forward contract. When the RHS is considered as a function of $P = \tilde{S}_1(X,t)$ for each t, it is recognized as the negative of the Legendre transform of the convex function $\tilde{S}(X,t)$. The LHS of (55) is clearly the product of the de-annualized variance rate $\eta^2\left(\frac{X}{\sqrt{M-t}}\right)(M-t)$ at time t and the stock's gamma $\tilde{S}_{11}(X,t)$ at time t. In this subsection, we provide a probabilistic interpretation of each side of (55).

The $\mathbb{Q}$ dynamics of the excess X are given by:

$$dX_t = \eta\left(\frac{X_t}{\sqrt{M-t}}\right)dW_t, \qquad t \in [0, M). \tag{56}$$

Clearly X is a $\mathbb{Q}$ martingale. Now, let:

$$S_t = \tilde{S}(X_t, t), \qquad t \in [0, M) \tag{57}$$

be the stock price process. Since $S_t = E_t^{\mathbb{Q}} X_M^+$, S is also a $\mathbb{Q}$ martingale. By the martingale representation theorem:

$$S_M = S_0 + \int_0^M P_t dX_t, \tag{58}$$

where $P_t = \tilde{S}_1(X_t, t)$ is the stock's delta w.r.t. asset value. Using (57) and integration by parts:

$$\tilde{S}(X_M, M) = \tilde{S}(X_0, 0) + \tilde{S}_1(X_M, M)X_M - \tilde{S}_1(X_0, 0)X_0 - \int_0^M X_t bP_t, \tag{59}$$

where bP signifies the use of a backward Itô integral. Re-arranging implies:

$$\begin{aligned}\tilde{S}_1(X_0,0)X_0 - \tilde{S}(X_0,0) &= \tilde{S}_1(X_M,M)X_M - \tilde{S}(X_M,M) - \int_0^M X_t bP_t \\ &= -\int_0^M X_t bP_t.\end{aligned} \tag{60}$$

since $\tilde{S}_1(X,M) = 1(X>0)$ and $\tilde{S}(X,M) = X^+ = X1(X>0)$. The RHS of (60) is the negative of the cost of rebalancing the delta hedge when replicating equity using forward contracts. Although

P is a stochastic process, the stochastic integral on the RHS of (60) is non-random. Negating both sides of (60) implies:

$$-[\tilde{S}_1(X_0,0)X_0 - \tilde{S}(X_0,0)] = \int_0^M X_t b P_t = \eta^2\left(\frac{X_0}{\sqrt{M}}\right) M\tilde{S}_{11}(X,0), \tag{61}$$

from evaluating (55) at $t=0$. In words, the amount lent at time 0 when replicating equity using forward contracts is both the non-random future cost of rebalancing the delta hedge and the product of the initial de-annualized variance rate $\eta^2\left(\frac{X_0}{\sqrt{M}}\right) M$ and the initial stock gamma $\tilde{S}_{11}(X_0,0)$.

Our closed form formula (54) for the equity value can be used to generate closed form formulas for equity greeks. Differentiating (54) w.r.t. A gives the stock delta:

$$\frac{\partial}{\partial A}S(A,t;L,M) = s'\left(\frac{A-L}{\sqrt{M-t}}\right) = \int_{-\infty}^{z} \frac{e^{-\int_0^y \frac{x}{\eta^2(x)}dx}}{b_s\eta^2(y)}dy \qquad A\in\mathbb{R}, t\in[0,M), \tag{62}$$

from (52) and (37). Thus, the stock delta $p(z) = \frac{\partial}{\partial A}S(A,t;L,M)$ is just a function of distance to default z. Since the stock delta p is increasing in z, it follows that there exists a function $\hat{z}(p)$ which is increasing in p. It follows that the assets' dollar volatility just depends on the stock delta in our class of models:

$$dA_t = \hat{\eta}(P_t)dW_t, \qquad t\in[0,M), \tag{63}$$

where $P_t = p\left(\frac{A_t-L}{\sqrt{M-t}}\right)$ is the stock delta at time t and $\hat{\eta}(p) \equiv \eta(\hat{z}(p))$ is the assets' dollar volatility, written as a function of the stock delta. Similarly, in the next section, we show that the assets' dollar volatility just depends on the normalized stock price in our class of models.

Finally, notice from (51) and (30) that the normalized stock price s and the failure probability f are each just a function of distance to default z. Since the failure probability f is decreasing in z, it follows that there exists a function $\tilde{z}(f)$, which is decreasing in f. Since the normalized stock price is an increasing function of only z, it follows that the normalized stock price is also some decreasing function $\tilde{s}(f) = s(\tilde{z}(f))$ of only f. In the section after next, we examine the inverse problem, i.e. determining the failure probability as a function of the normalized stock price.

4 Calibrating Local Volatility to Equity Call Prices

In this section, we assume that the asset volatility function $\eta(z)$ is not known *ex ante*. To make up for this informational shortfall, we assume that one can observe the market price of the stock. We also assume that there exists a liquid market in calls written on the stock that mature before the debt. These equity calls are compound calls written on the assets of the firm.

We assume that market prices of the compound calls can be directly observed. Since these compound calls on the assets are nothing but vanilla calls on the equity, this assumption is realistic. Recall that our assumption (18) implies that the normalized stock price $s_t = \frac{S_t}{\sqrt{M-t}}$ is a function $s(z)$ of just the distance to default process $z_t = \frac{A_t-L}{\sqrt{M-t}}$. Since $s(z)$ is increasing in z, it follows that there exists a function $z(s)$ which is increasing in s. Substituting into (18) implies that the assets' dollar volatility just depends on the normalized stock price $s_t = \frac{S_t}{\sqrt{M-t}}$ in our class of models:

$$dA_t = \eta(z(s_t))dW_t, \qquad t \in [0, M). \tag{64}$$

From (62), the stock's delta, $\frac{\partial S}{\partial A}$, depends on the four variables A_t, t, L, and M only through the distance to default variable $z_t = \frac{A_t - L}{\sqrt{M-t}}$. Let $s'(z)$ be the stock's delta w.r.t. assets. Using (64), an application of Itô's formula to $S_t = S(A_t, t; L, M)$ implies:

$$dS_t = s'(z(s_t))\eta(z(s_t))dW_t = a(s_t)dW_t, \qquad t \in [0, M),$$

where:

$$a(s) \equiv s'(z(s))\eta(z(s)), \qquad s > 0, \tag{65}$$

is referred to as the local volatility function. Since the stock's volatility just depends on the stock's price and time, the stock price dynamics are described by a (special case of a) local volatility model. Since $\eta(z)$ is unknown ex ante, the local volatility function $a(s)$ is also unknown ex ante.

Let $C_0(K, T)$ be the market price at time 0 of a call written on the equity struck at some $K > 0$ and maturing at some $T \in [0, M]$. At its maturity date T, the call pays $[S(A_T, T; L, M) - K]^+$ dollars to its holder. Since the stock price is following a diffusion process, the existence of the maturity derivative $\frac{\partial}{\partial T} C(S, t; K, T; M)$ is guaranteed. This diffusion also causes the observed call pricing function to be twice differentiable in strike, i.e. $\frac{\partial^2}{\partial K^2} C(S, t; K, T; M)$ exists. The results of Dupire[5] imply that call prices solve the following forward PDE:

$$\frac{1}{2} a^2 \left(\frac{K}{\sqrt{M-T}} \right) \frac{\partial^2 C_0}{\partial K^2}(K, T) = \frac{\partial C_0}{\partial T}(K, T), \qquad K > 0, \tag{66}$$

subject to the initial condition:

$$C_0(K, 0) = (S - K)^+, \qquad K > 0. \tag{67}$$

We now assume that one can observe both market call prices $C_0(K, T)$ and infinitessimal calendar spreads $\frac{\partial}{\partial T} C_0(K, T)$ for a continuum of strikes $K > 0$ at the fixed call maturity date $T \in [0, M]$. As a result, the local variance rate can be observed:

$$a^2 \left(\frac{K}{\sqrt{M-T}} \right) = \frac{2 \frac{\partial}{\partial T} C_0(K, T)}{\frac{\partial^2}{\partial K^2} C_0(K, T)}, \qquad K > 0. \tag{68}$$

5 Formulas for Implied RNDP and Implied Asset Value

The objective of this section is to use the local volatility function to determine the implied RNDP and the implied asset value as functions of the stock price and time to maturity of the debt. We will show that both functions can be explicitly determined once a local volatility smile has been obtained from the options market. Hence, when the firm's stock price is observed and the debt maturity date is known, the RNDP and distance to default can be evaluated. The functions can also be used to assess RNDP and distance to default in various stock and option price scenarios.

5.1 Formula for Implied RNDP

In this subsection, we suppose that the debt level L and the asset value A cannot be directly observed. We will nonetheless show that the failure probability can be calculated. Recall from (28) that the function $f(z)$ relating the failure probability F to distance to default $z \equiv \frac{A-L}{\sqrt{M-t}}$ solves the ODE:

$$\eta^2(z) f''(z) + z f'(z) = 0, \qquad z \in \mathbb{R}. \tag{69}$$

Let $\phi(s) \equiv f(z(s))$ be a change of dependent variable. Since $f(z) = \phi(s)$, differentiating w.r.t. z implies:

$$f'(z) = \phi'(s) s'(z), \qquad z \in \mathbb{R}. \tag{70}$$

Differentiating (70) w.r.t. z implies:

$$f''(z) = \phi''(s)(s'(z))^2 + \phi'(s) s''(z), \qquad z \in \mathbb{R}. \tag{71}$$

Substituting (70) and (71) in (69) yields:

$$\eta^2(z)(s'(z))^2 \phi''(s) + \left[\eta^2(z) s''(z) + z s'(z)\right] \phi'(s) = 0, \qquad s > 0. \tag{72}$$

Now recall from (42) the following second order linear ODE for the normalized stock price $s(z)$:

$$\eta^2(z) s''(z) + z s'(z) - s(z) = 0, \qquad z \in \mathbb{R}. \tag{73}$$

Thus, the coefficient on $\phi'(s)$ in (72) is simply s. Furthermore, the coefficient on $\phi''(s)$ in (72) is simply $a^2(s)$ from (65). As a result, the function relating failure probability to normalized stock price is governed by the following simple second order linear ODE:

$$a^2(s)\phi''(s) + s\phi'(s) = 0, \qquad s > 0. \tag{74}$$

Comparing (74) with (28) implies that the failure probability solves the same type of ODE as before, but where the leading coefficient depends on the observable volatility function $a(s)$, instead of the unobserved volatility function $\eta(z)$.

Suppose that we impose the following Dirichlet boundary conditions for $\phi(s)$:

$$\lim_{s \downarrow 0} \phi(s) = 1 \text{ and } \lim_{s \uparrow \infty} \phi(s) = 0. \tag{75}$$

Then the unique solution to this two point boundary value problem is:

$$\phi(s) = 1 - \frac{1}{b_\phi} \int_0^s e^{-\int_0^y \frac{z}{a^2(z)} dz} dy, \qquad s > 0, \tag{76}$$

where the positive normalizing constant b_ϕ is given by:

$$b_\phi = \int_0^\infty e^{-\int_0^y \frac{z}{a^2(z)} dz} dy. \tag{77}$$

Thus, the failure probability is given by the following explicit function:

$$F(A(S,t;L,M),t;L,M) = 1 - \frac{1}{b_\phi} \int_0^{\frac{S}{\sqrt{M-t}}} e^{-\int_0^y \frac{z}{a^2(z)} dz} dy, \qquad S > 0, t \in [0,T), \tag{78}$$

where b_ϕ is given in (77). Notice that the RHS of (78) does not depend on the time t asset value A or on the face value L of the debt. Instead, the failure probability only depends on the time t stock price S, the time to the debt maturity $M-t$, and the local volatility function $a^2(z)$ determined from the initial call prices of maturity $T \in (0,M)$. These variables represent sufficient statistics for the determination of the RNDP. In the Merton model, knowledge of the stock price obviates the need to observe asset value for the purpose of calculating the failure probability. However, one still needs knowledge of the debt level L.

5.2 Formula for Implied Asset Value

In the last subsection, we inverted $\hat{s}(f)$ to arrive at an explicit formula for $\phi(s)$. We now seek to derive an explicit formula for the implied asset value as a function of the stock price and the time to the debt maturity.

We will accomplish this task by first establishing an explicit formula relating distance to default $z \equiv \frac{A-L}{\sqrt{M-t}}$ to the normalized stock price $s \equiv \frac{S}{\sqrt{M-t}}$. Recall that we developed an explicit formula (51) in section 3.2 relating s to z when the asset volatility function $\eta(z)$ is known. We now switch attention to the problem of explicitly inverting this function $s(z)$. Let $z(s), s > 0$ denote the desired inverse. We seek an explicit formula for $z(s), s > 0$ when the local volatility function $a(s)$ is known.

We try a hodograph transformation on the ODE (73), i.e. we consider the effect of switching the roles of the dependent and independent variables. By the inverse function theorem, we have:

$$s'(z) = \frac{1}{z'(s)}, \qquad s > 0. \tag{79}$$

Differentiating (79) w.r.t. z implies:

$$s''(z) = -\frac{z''(s)}{[z'(s)]^3} = -(s'(z))^2 \frac{z''(s)}{z'(s)}, \qquad s > 0, \tag{80}$$

from (79). Substituting (79) and (80) in the linear ODE (73) leads to the non-linear ODE:

$$-\eta^2(z(s))(s'(z(s)))^2 \frac{z''(s)}{z'(s)} + \frac{z(s)}{z'(s)} - s = 0, \qquad s > 0. \tag{81}$$

However, substituting (65) in (81) and multiplying by $-z'(s)$ leads to the following linear second order ODE:

$$a^2(s)z''(s) + sz'(s) - z(s) = 0, \qquad s > 0. \tag{82}$$

Comparing (82) to (73), we see that (73) is partially invariant to a hodograph transformation. The coefficient on the second derivative changes and the domain changes from $\mathbb{R}$ to $\mathbb{R}^+$, but the remainder of the ODE is the same.

Using the same logic that lead to (45), the general solution to the ODE (82) has the form:

$$g(s) = B_1 s + B_2 s \int_{s_0}^{s} \frac{e^{-\int_{s_0}^{y} \frac{x}{a^2(x)} dx}}{y^2} dy, \qquad s > 0, \tag{83}$$

where B_1 and B_2 are constants and where $s_0 = \frac{S_0}{\sqrt{M}}$ is the initial normalized stock price. Setting $s = s_0$ in (83) implies that $g(s_0) = B_1 s_0$. In contrast to the last subsection, we now assume we can directly observe the initial distance to default $z_0 = \frac{A_0 - L}{M}$. Hence, we know one value of the function $z(s)$:

$$z(s_0) = z_0. \tag{84}$$

Setting $s = s_0$ in (83) and imposing (84) forces $B_1 = \frac{z_0}{s_0}$ and hence:

$$g(s) = \left(\frac{z_0}{s_0} + B_2 \int_{s_0}^{s} \frac{e^{-\int_{s_0}^{y} \frac{x}{a^2(x)} dx}}{y^2} dy \right) s, \qquad s > 0. \tag{85}$$

We can now determine the constant B_2 in (85) by imposing the following Dirichlet upper boundary condition:

$$\lim_{s \uparrow \infty} z(s) \sim s. \tag{86}$$

Let:

$$b_z = \int_{s_0}^{\infty} \frac{e^{-\int_{s_0}^{y} \frac{x}{a^2(x)} dx}}{y^2} dy. \tag{87}$$

Hence (85) implies that:

$$\lim_{s \uparrow \infty} g(s) \sim \left(\frac{z_0}{s_0} + B_2 b_z \right) s. \tag{88}$$

In order to satisfy the upper boundary condition (86), we must have:

$$B_2 = \frac{1 - \frac{z_0}{s_0}}{b_z}. \tag{89}$$

Therefore, our final solution for $z(s)$ is:

$$z(s) = \left(\frac{z_0}{s_0} + \frac{1 - \frac{z_0}{s_0}}{b_z} \int_{s_0}^{s} \frac{e^{-\int_{s_0}^{y} \frac{x}{a^2(x)} dx}}{y^2} dy \right) s, \qquad s > 0. \tag{90}$$

Integration by parts allows us to write the following:

$$\int_{s_0}^{s} \frac{e^{-\int_{s_0}^{s} \frac{x}{a^2(x)}dx}}{y^2} dy = -\left.\frac{e^{-\int_{s_0}^{y} \frac{x}{a^2(x)}dx}}{y}\right|_{y=s_0}^{y=s} - \int_{s_0}^{s} \frac{e^{-\int_{s_0}^{y} \frac{x}{a^2(x)}dx}}{a^2(y)} dy \tag{91}$$

$$= \frac{1}{s_0} - \frac{e^{-\int_{s_0}^{s} \frac{x}{a^2(x)}dx}}{s} - \int_{s_0}^{s} \frac{e^{-\int_{s_0}^{y} \frac{x}{a^2(x)}dx}}{a^2(y)} dy, \qquad s > 0. \tag{92}$$

As a result, our formula may be re-written as:

$$z(s) = \frac{z_0}{s_0} s + \frac{1 - \frac{z_0}{s_0}}{b_z} s \left(\frac{1}{s_0} - \frac{e^{-\int_{s_0}^{s} \frac{x}{a^2(x)}dx}}{s} - \int_{s_0}^{s} \frac{e^{-\int_{s_0}^{y} \frac{x}{a^2(x)}dx}}{a^2(y)} dy \right) \tag{93}$$

$$= \frac{s}{s_0} \left(z_0 + \frac{1 - \frac{z_0}{s_0}}{b_z} \right) - \frac{1 - \frac{z_0}{s_0}}{b_z} \left(e^{-\int_{s_0}^{s} \frac{x}{a^2(x)}dx} + s \int_{s_0}^{s} \frac{e^{-\int_{s_0}^{y} \frac{x}{a^2(x)}dx}}{a^2(y)} dy \right), \qquad s > 0. \tag{94}$$

Taking the derivative w.r.t. s results in a fortuitous cancellation, yielding:

$$z'(s) = \frac{1}{s_0} \left(z_0 + \frac{1 - \frac{z_0}{s_0}}{b_z} \right) - \frac{1 - \frac{z_0}{s_0}}{b_z} \int_{s_0}^{s} \frac{e^{-\int_{s_0}^{y} \frac{x}{a^2(x)}dx}}{a^2(y)} dy, \qquad s > 0. \tag{95}$$

To see that $z'(s)$ is positive for all $s > 0$, first observe that this is clearly true for $s < s_0$ since both terms in (95) are positive in that range. Next, observe that differentiating (95) with respect to s implies that:

$$z''(s) = -\frac{1 - \frac{z_0}{s_0}}{b_z} \frac{e^{-\int_{s_0}^{s} \frac{x}{a^2(x)}dx}}{a^2(s)}, \qquad s > 0. \tag{96}$$

Since S_0 is the value of a call option, it has nonnegative time value and hence

$$S_0 > A_0 - L. \tag{97}$$

Dividing by $\sqrt{M}$ implies $\frac{z_0}{s_0} < 1$. Hence, from (96), $z''(s)$ is negative. Therefore, $z(s)$ is concave and $z'(s)$ is decreasing. Finally, by (92) we see that:

$$\lim_{s \uparrow \infty} \int_{s_0}^{s} \frac{e^{-\int_{s_0}^{y} \frac{x}{a^2(x)}dx}}{a^2(y)} dy = \frac{1}{s_0} - b_z, \tag{98}$$

and so:

$$\lim_{s\uparrow\infty} z'(s) = 1. \tag{99}$$

Combining this limit with the fact that $z'(s)$ is decreasing and $z'(s) > 0$ for $0 < s < s_0$ allows us to conclude that $z'(s) > 0$ for all $s > 0$ and hence $z(s)$ is increasing.

With the function $z(s)$ given by the simpler formula (90), we can easily determine the implied asset value:

$$A(S, t; L, M) = L + \sqrt{M-t}\,z(S/\sqrt{M-t}). \tag{100}$$

This function gives the market-consistent asset value as a function of stock price S for each t. Evaluating this function at the observed current stock price recovers the implied contemporaneous asset value used to calculate z_0 in (84). If a second set of call maturities are available, one can test the model by seeing if the revealed asset value is the same for the two maturities. By numerically evaluating the function $A(S, t; L, M)$ for whatever range of stock values $S \in (\ell, h)$ are of interest, one gets the function $S(A, t; L, M)$ relating equity value to asset value for this range of equity values.

Summarizing our progress to date, the local volatility smile obtained from stock option prices has been used to analytically invert both $\hat{s}(f)$ and $s(z)$. As a result, the implied RNDP and the implied asset value have both been explicitly expressed in terms of the initial stock price and the time to the debt maturity. It is not necessary to determine the assets' volatility function to determine these fundamental quantities, but we now nonetheless turn our attention to the determination of this variable.

Let $s_a(z)$ be the inverse of the function $z(s)$ given by (90). The subscript a reminds us that this function differs from the function $s(z)$ obtained when $\eta(z)$ is known. Our new function $s_a(z)$ is obtained by numerical inversion of $z(s)$. One numerically evaluates $z(s)$ for whatever range of normalized stock values s are of interest. One then reads the two vectors in reverse order to get the function $s_a(z)$ relating normalized stock price to distance to default. From the definition of normalized stock price, one obtains the following stock valuation function:

$$S_a(z, t; M) = \sqrt{M-t}\,s_a(z), \qquad z \in \mathbb{R}. \tag{101}$$

To instead obtain the asset volatility function from $s_a(z)$, note that substituting (79) in (65) implies:

$$a(s) = \frac{\eta(z(s))}{z'(s)}, \qquad s > 0. \tag{102}$$

Re-arranging and evaluating at $s = s_a(z)$ implies that:

$$\eta(z) = z'(s_a(z))a(s_a(z)), \qquad z \in \mathbb{R}. \tag{103}$$

Thus, the function η relating the asset's dollar volatility to distance to default z can be numerically determined in a manner consistent with call prices and calendar spreads at a single maturity. As this function was the original starting point of the analysis in this paper, we have come full circle. Recall that the function $\eta(z)$ can also be used to numerically determine how management chooses the firm's asset volatility at each level of the RNDP. Hence, one can empirically determine from stock and stock option prices whether management's interests are more aligned with the interests of bondholders or the interests of shareholders. It can also be used to assess the impact of a managerial compensation plan on these incentives.

6 Summary and Future Research

In his classic paper, Merton assumed that the normal volatility of a firm's assets is positively proportional to the value of the firm's assets. In this paper, we instead assume that normal volatility of a firm's assets is a function of the risk-neutral default probability (RNDP). By modeling the function relating asset volatility to distance to default, we are able to give a closed form formula for both the RNDP and for the equity of the firm. To our knowledge, our explicit valuation formulae are the first to be derived when volatility is free to be any function, rather than described by a particular functional form with a small number of free parameters. The flexibility of our specification allows the RNDP function and the equity valuation function to be consistent with the market prices of calls written on the stock. We show how to analytically calculate the stock's local volatility function as a functional of the market prices of calendar spreads and butterfly spreads. We then analytically relate the implied RNDP and the implied asset value to the stock price and the local volatility smile. The RNDP formula is independent of the asset value A and the promised debt payment L. The formula just depends on the local volatility smile $a(S), S > 0$, the stock price S, and the time to the debt maturity $M - t$.

One future research direction is to numerically implement the formulas. Another future research direction is to try to explicitly value other asset value derivatives given the asset volatility function $\eta(z)$. For example, suppose $\eta(z)$ is symmetric in z. Consider a one touch written on the firm's assets with lower barrier L and maturity date M. The reflection principle implies that this one-touch is valued at twice the failure probability, so one has a closed form formula. One can also try to value compound calls on the asset value, given the asset volatility function $\eta(z)$, which need not be symmetric. A related problem is to value equity when the debt receives discrete coupons and/or there are multiple maturities. Similarly, one can value equity when a financial intermediary can default. When the counterparty cannot default, the dollars in the riskfree asset (aka exposure) when replicating equity using forward contracts is $S - (A - L)\frac{\partial S}{\partial A}$, which is positive, since it equals $\eta^2(\frac{A-L}{\sqrt{M-t}})(M - t)\frac{\partial^2 S}{\partial A^2} = 2(M - t)\frac{\partial S}{\partial M}$. Suppose that the risk-neutral arrival rate of counterparty default is $\frac{\lambda}{(A-L)^2}$ where λ is positive. Then the Brownian scaling property still holds so that the equity can still be valued in closed form. Similarly, when $s = \frac{S}{\sqrt{M-t}}$ is the state variable rather than $z = \frac{A-L}{\sqrt{M-t}}$, an arrival rate of the form $\frac{\lambda}{s^2}$ is tractable with exposure equal to $z - s\frac{\partial z}{\partial s}$.

A third research direction invokes exploring other measures of the distance to default. A generalization of the Bachelier model that maintains a tradeoff between space scaling and time scaling is the CEV model. Although typically applied to describe nonnegative processes, the CEV model can also describe real-valued processes through symmetrization. The Merton model arises as a special case of the CEV framework, but is the one member of the CEV class that destroys the tradeoff between space scaling and time scaling. Bessel processes also enjoy the Brownian scaling property which we exploited. Powers of Bessel processes also enjoy scaling and the CEV martingale is just a special case. Another class of processes which allow a tradeoff between space scaling and time scaling is anomalous diffusion. The transition density for this process is called a q gaussian density and is given in closed form by Tsallis and others. This class of processes also contains standard Brownian motion as a special case.

A fourth research direction would be to investigate which volatility functions allow the integrals to converge and when they do, allow the integrals to be expressed analytically or in terms of special functions. For example, when valuing the stock, we know that setting $\eta(z)$ to a constant leads to the Bachelier call formula, where the integral describing delta can be expressed using the standard normal distribution function.

For a fifth research direction, one can allow the stock to receive dividends. If the dividends are paid continuously and just depend on distance to default, it should still be possible to derive closed form valuation formulas for both failure probability and equity value. For a sixth research direction, one can add a jump to default of the assets. When the arrival rate of the jump is inversely proportional to the square of the net asset value, the Brownian scaling property is preserved. For a seventh research direction, note that in the Merton model, default can occur only at expiry. In contrast, Black and Cox[2] model default as occurring at the first time that the firm's asset value hits some flat barrier. Thus, it would be interesting to try to extend our results to the case where assets can jump and default occurs at the first time that A is at or below the barrier L. One would be given the normal volatility function $\eta(z)$ for $z \geq 0$. By working with its even extension, the reflection principle should allow closed form valuation of equity when it is considered as a down-and-out call on the assets of the firm.

A seventh research direction is to use Skorohod's lemma to expand the state space from z to e.g. $(\underline{z}, z^v)$ or $(L^z, |z|)$. When $\eta(z)$ is even, one should be able to transfer knowledge of the conditional law of z_M, evaluated at $z_M = 0$ to a bivariate conditional law. One can then use ($\underline{z}$ and z^v) to create a positive martingale to describe asset value. Hopefully, the failure probability can still be calculated analytically.

An eighth research direction is to instead expand the state space by adding either instantaneous asset variance or implied stock variance as a state variable. When some notion of volatility is stochastic, it has been noted that one can use the Black Scholes formula to correct its own assumption of a constant volatility parameter. One can similarly try to correct the current model's assumption of a constant volatility function.

A ninth research direction is to replace the local volatility smile as a calibration instrument with the term structure of credit default swaps (CDS). Alternatively, one can expand the model so that it can be calibrated to both a local volatility smile and a CDS term structure. A tenth research direction would be to consider the control problem where management chooses to maximize the real world expected value of growth in the stock price i.e. $E_0^{\mathbb{P}} \ln S(A_t, t; L, T)$ by choice of the volatility function $\eta(z)$. Thinking of arithmetic distance to default as the state variable, increases in volatility have the advantage of increasing expected return $E_0^{\mathbb{P}} \frac{dS(A_t,t;L,T)}{S(A_t,t;L,T)}$, but have the disadvantage of increasing variance, $\left(\frac{dS(A_t,t;L,T)}{S(A_t,t;L,T)}\right)^2$.

One can also consider adding stochastic interest rates or at least asset level-dependent interest rates. When the riskfree rate is inversely proportional to the square of the net asset value, the Brownian scaling property is preserved.

Reverting to zero interest rates and dividends, one can explore whether the stock's local volatility function $a(\cdot)$ can be numerically determined from just the call price data $C_0(K), K > 0$ without knowledge of the calendar spread data $\frac{\partial}{\partial T} C_0(K), K > 0$.

A thirteenth research direction would explore the connection of this work to Legendre transforms. The normalized equity value function $s(z)$ is convex in z and solves an ODE which equates the product of instantaneous variance $\eta^2(z)$ and gamma $s_{zz}(z)$ to the negative of its Legendre transform $-s^*(p)$, where $p = s_z(z)$. It can be shown that the second derivative of this Legendre transform is just the reciprocal of gamma, i.e. $s^*_{pp}(p) = \frac{1}{s_{zz}(z)}$. Hence, the geometric mean of $|s^*(p)|$ and $s^*_{pp}(p)$ is the instantaneous volatility $\eta(z(p))$. Hence the Legendre transform naturally emphasizes the connection between (instantaneous asset dollar) volatility and (stock) delta. This is roughly consistent with the FX option convention of quoted (Black Scholes implied) volatility as a function of (Black Scholes) delta.

Finally, the empirical implications of the class of models should be tested. In the interests of brevity, these fourteen extensions are best left for future research.

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